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THEORETICAL STUDY OF NON-STANDARD IMAGING CONCEPTS. VOLUME I

David L. Fried

Optical Science Consultants

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THEORETICAL STUDY OF NON-STANDARD IMAGING CONCEPTS

Optical Science Consultants

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APPROVED: Darryl P. Greenwood

DARRYL P. GREENWOOD, Capt, USAF
Project Engineer

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This report presents the most significant of the results developed under this contract during the period July 1974 to February 1975 on the subject of non-standard imaging concepts. The work reported here represents a complete treatment from development of the theory to presentation and interpretation of numerical results for the operation of a CENSORING System. The question asked is essentially what is the probability that at any instant of the random wavefront distortion over some circular aperture will be close enough to nothing more complex than a tilted plane, that a short exposure image formed at that

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instant will be almost diffraction-limited. (The problem is perhaps most succinctly defined by the question -- How many pictures do you have to take to get a good one?) The numerical results show that the probability is an exponential function of aperture area divided by r_0^2. If $D/r_0 = (7, 10, 15)$, the probability of getting a nearly diffraction-limited image is found to be $P_{CENSOR} \approx (3 \times 10^{-3}, 1 \times 10^{-6}, 3.4 \times 10^{-15})$. The functional relationship is

$$P_{CENSOR} \approx 5.6 \exp [-0.1557 (D/r_0)^2]$$

Derivation and basic results are presented in this main volume. Certain of the more voluminous tables are presented in a separate addendum volume.

THEORETICAL STUDY OF NON-STANDARD IMAGING CONCEPTS

Dr. David L. Fried

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Phone: 714 524-3622

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Phone: 315 330-3145

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ABSTRACT

In Part I of this report, the performance of a CENSORING system is examined from the point of view of determining the probability, P_{Censor}, that at any instant of time the random wavefront distortion over a circular aperture of diameter D will be small enough to allow a nearly diffraction-limited image to be formed. It is pointed out that the effective wavefront distortion for this calculation is not the total distortion function, $\phi(\vec{r})$, but rather $\varphi(\vec{r};D)$, which represents the distortion after subtraction of the instantaneous average phase and tilt over the aperture. The problem of calculating the probability is related to the probabilities for the value of the various random coefficients β_n of the decomposition of $\varphi(\vec{r};D)$ into a series $\sum_n \beta_n f_n(\vec{r};D)$, where the functions f_n are orthonormal functions over the aperture chosen so as to make the various β_n statistically independent. (It is noted that the β_n are gaussianly distributed, since φ is a gaussian random function.) The appropriate Karhunen-Loeve homogeneous integral equation is developed to allow the f_n to be obtained as eigenfunctions. The variance of the β_n are seen to be the corresponding eigenvalues. It is then shown how the probability P_{Censor} can be calculated as a multi-dimensional integral over the product of gaussian distribution with the variances corresponding to those of the β_n. The numerical solution of the Karhunen-Loeve homogeneous integral equation to obtain the eigenvalues (and the eigenfunctions) and the numerical evaluation of the multi-dimensional integral which finally gives P_{Censor} are taken up in Part II of this report.

In Part II of this report, the eigenvalues and the eigenfunctions for average tilt and average phase suppressed wavefront distortion on a circular aperture are developed. (The eigenvalues and eigenfunctions without tilt distortion suppressed are also developed.) Using the eigenvalue set, the CENSORING system probability, P_{CENSOR}, of getting a short exposure

image with less than one radian squared distortion averaged over the aperture is evaluated as a multi-dimensional integral, evaluated by Monte Carlo techniques. The results are found to have the form

$$P_{CENSOR} \approx 5.6 \exp[-0.1557 (D/r_0)^2] \quad .$$

The exponential dependence on aperture area is in agreement with an earlier conjecture by Hufnagel (though with significantly different coefficients than were suggested by Hufnagel).

It is noted that this exponential dependence on aperture diameter makes the proper selection of D/r_0 a very critical aspect of a CENSORING experiment. It is pointed out that if independent samples of wavefront distortion are obtained every 10 msec, and $D/r_0 = 15$ (corresponding to $D = 1.5$ m, $r_0 = 0.1$ m), then we would have to wait about 800 million hours for a good picture. If D/r_0 is reduced to a value of 10 (corresponding to $D = 1.0$ m, $r_0 = 0.1$ m), the waiting time is reduced to about 2.5 hours, while if D/r_0 is reduced to 7 (corresponding to $D = 0.7$ m, $r_0 = 0.1$ m), the waiting time drops to only 3.5 seconds.

The report has been bound in two volumes. The main volume presents the derivation and principal results. The addendum volume presents the more voluminous tables of intermediate results, particularly those that may be of use in working other problems.

PART I

Formal Theory of CENSORING Systems Operation

Introduction

The concept of CENSORING as a method of obtaining high resolution images through atmospheric turbulence is based on the assumption that of all possible forms that random wavefront distortion will assume during some period of time, there is a finite probability that at some instant the random pattern will very nearly resemble a plane wave. At that instant, a nearly diffraction-limited image can be obtained. A CENSORING system would be able to recognize this condition quickly enough to allow a photograph to be taken just then, while preventing exposures at times of more normal distortion.

The simplicity of the CENSORING concept makes it seem quite attractive, but on the other hand it is entirely dependent on random occurrences for its operation. It is therefore critical that we understand the probabilities involved and be able to estimate the time we will have to wait, on an average, before the CENSORING system will provide us with a picture. It appears likely that the probability that at any instant of time the distorted wavefront will be reasonably close to a plane wave is inversely proportional to the telescope aperture area (divided by r_0^2) so that there is a practical limit of useful telescope size for a CENSORING system. The larger the telescope diameter, the longer we have to wait for a good picture, with waiting time increasing dramatically as telescope size is increased.

For this reason, we desire a quantitative understanding of the probabilities involved in a CENSORING system's operation. This paper is aimed at the formulation of that theory. Here we shall be concerned with setting up the basic formulations and deriving equations suitable for computer evaluation. In a later paper, we shall carry out the necessary computer calculations.

Wavefront Distortion Analysis

The key to analysis of the wavefront distortion probabilities involved in CENSORING operation is the decomposition of a sample of the random wavefront taken over the aperture into a set of orthonormal functions whose coefficients in the decomposition series representation are independent random variables. From knowledge of the mean-square value of these random coefficients, we can calculate the appropriate probabilities for CENSORING system operation.

The orthonormal decomposition with independent random coefficients is related to the Karhunen-Loeve theorem and we can anticipate that the development of the orthonormal functions and the evaluation of the mean-square value of the coefficients will depend on the solution of a homogeneous integral equation for its eigenfunctions and eigenvalues, respectively. The key to that effort is the development of the kernel for the integral equation. The kernel is developed from the statistics of wavefront distortion and, as we may naturally expect, is related to the phase structure function. However, as we shall see, the relationship is by no means trivial and requires careful development.

In order to calculate the kernel, we first have to define in exact terms the nature of the wavefront distortion statistics and the "portion" of the distortion that is of concern to us. We denote the random wavefront distortion (measured in radians of phase) at a point $\vec{r}$ on the aperture plane by $\phi(\vec{r})$. Over a circular aperture of diameter D a random sample of the distorted wavefront has a random average phase $\bar{\phi}$, and a random average tilt $\vec{\alpha}$, where

$$\bar{\phi} = (\tfrac{1}{4}\pi D^2)^{-1} \int d\vec{r}\; W(\vec{r}, D)\, \phi(\vec{r}) \quad , \qquad (1)$$

and

$$\vec{\alpha} = (\tfrac{1}{64}\pi D^4)^{-1} \int d\vec{r}\; W(\vec{r}, D)\, \vec{r}\, \phi(\vec{r}) \quad , \qquad (2)$$

where $W(\vec{r}, D)$ is an aperture function defining, in the $\vec{r}$-plane, a circle of diameter D centered at the origin. This aperture function is defined by the equation

$$W(\vec{r}, D) = \begin{cases} 1 \ , & \text{if } |\vec{r}| \leq \frac{1}{2} D \\ 0 \ , & \text{if } |\vec{r}| > \frac{1}{2} D \end{cases} \ , \tag{3}$$

so that in effect, it defines the limits of integration for the $\vec{r}$-integration in Eq.'s (1) and (2), which are otherwise taken to be over the infinite $\vec{r}$-plane. The normalization in Eq. (2) has been chosen so that if $\phi(\vec{r}) \equiv \vec{a} \cdot \vec{r}$, then we would obtain from Eq. (2) the relationship $\vec{\alpha} = \vec{a}$.

We note that in forming a short-exposure image, neither the average phase, $\bar{\phi}$, nor the wavefront tilt, $\vec{\alpha}$, disturb the resolution of the image. The tilt produces an image shift which is basically a nonobservable in the sense that without special effort to provide an absolute angular orientation reference, the effect of the tilt will not be measurable. Thus, from our point of view, the effective instantaneous random wavefront distortion over the aperture is

$$\varphi(\vec{r};D) = \phi(\vec{r}) - \bar{\phi} - \vec{\alpha} \cdot \vec{r} \quad . \tag{4}$$

If φ is small enough, then the image will be nearly diffraction-limited no matter how large $\bar{\phi}$ and $\vec{\alpha}$ are. We wish to calculate the probability distribution for φ as the basis for determining the statistics of the operation of a CENSORING system.

To provide the basis for our calculations of these statistics, at this point we introduce the set of functions, $\{f_n(\vec{r};D)\}$ with the orthonormal property that

$$\int d\vec{r}\ W(\vec{r}, D)\, f_n^*(\vec{r};D)\, f_{n'}(\vec{r};D) = \begin{cases} 1 \ , & \text{if } n = n' \\ 0 \ , & \text{if } n \neq n' \end{cases} \tag{5}$$

and the completeness property that for any random sample $\varphi(\vec{r};D)$ we can write

$$\varphi(\vec{r};D) = \sum_n \beta_n f_n(\vec{r};D) \quad . \tag{6}$$

Where $\{\beta_n\}$ is an appropriately chosen set of coefficients. Because of the orthonormal property of f_n, as defined by Eq. (5), it follows from Eq. (6) that

$$\beta_n = \int d\vec{r}\, W(\vec{r}, D)\, f_n^*(\vec{r};D)\, \varphi(\vec{r};D) \quad . \tag{7}$$

Obviously, then, just as φ is a random function, β_n is a random variable. We also note that since ϕ is a gaussian random function, then $\overline{\phi}$ and $\vec{a}$, being linear functions of ϕ are gaussian random variables. From this, in turn, it follows that φ is a gaussian random function -- and this, in turn, implies that β_n, being a linear function of φ, is a gaussian random variable. This fact, together with our ability to calculate the variance of β_n (which we shall obtain as the eigenvalues of the integral equation defining f_n), will provide the basis of calculating the probability of φ taking a low enough wavefront distortion form.

The key to the definition of the set of orthonormal functions $\{f_n\}$ from all possible sets of functions that are orthonormal over the region defined by $W(\vec{r}, D)$ is the requirement that the various random coefficients β_n must be independent. We require that

$$\langle \beta_n^* \beta_{n'} \rangle = \begin{cases} B_n^2(D) \quad , & \text{if} \quad n = n' \\ 0 \quad , & \text{if} \quad n \neq n' \end{cases} \quad , \tag{8}$$

where $B_n^2(D)$ denotes the variance of the random variable β_n. (In writing B_n^2, we have chosen, as a matter of convenience for later work, to make the dependence of B_n^2 on the aperture diameter, D, explicit.) To see the

implications of Eq. (8), we consider the quantity

$$\mathcal{S} = \langle \int d\vec{r}' \; W(\vec{r}', D) \, \varphi^*(\vec{r}';D) \, \varphi(\vec{r}, D) \, f_n(\vec{r}';D)\rangle \quad . \tag{9}$$

We define the covariance of φ as

$$C_\varphi(|\vec{r} - \vec{r}'|;D) = \langle \varphi^*(\vec{r};D) \, \varphi(\vec{r}';D)\rangle \quad , \tag{10}$$

where the homogeneity and isotropy of the propagation statistics have allowed us to write the dependence on $\vec{r}$ and $\vec{r}'$ in the form of $|\vec{r} - \vec{r}'|$. We note that by interchanging the order of ensemble averaging and integration, we can rewrite Eq. (9) in the form

$$\mathcal{S} = \int d\vec{r}' \; W(\vec{r}', D) \, C_\varphi(|\vec{r} - \vec{r}'|;D) \, f_n(\vec{r}';D) \quad . \tag{11}$$

However, if we use Eq. (6) to provide a series representation for $\varphi^*(\vec{r}';D)$ to be substituted into Eq. (9), we get

$$\mathcal{S} = \langle \int d\vec{r}' \; W(\vec{r}', D) \, \varphi(\vec{r};D) \sum_{n'} \beta_{n'}^* \, f_{n'}^*(\vec{r}';D) \, f_n(\vec{r}';D)\rangle \; , \tag{12}$$

which on interchanging the order of integration and summation gives us

$$\mathcal{S} = \langle \varphi(\vec{r};D) \sum_{n'} \beta_{n'}^* \int d\vec{r}' \; W(\vec{r}', D) \, f_{n'}^*(\vec{r}';D) \, f_n(\vec{r}';D)\rangle \quad . \tag{13}$$

The orthonormal property of f_n as expressed in Eq. (5) allows the integration to be performed, and the result of this allows us to reduce the summation over n' to the single term for which $n' = n$. Thus we get

$$\mathcal{S} = \langle \varphi(\vec{r};D) \, \beta_n^*\rangle \quad . \tag{14}$$

Now if we again use Eq. (6) to provide a series representation for $\varphi(\vec{r};D)$, we get

$$\mathcal{S} = \langle \sum_{n'} \beta_{n'} f_{n'}(\vec{r};D) \beta_n^* \rangle \tag{15}$$

An interchange of the order of summation and ensemble averaging leads to the result

$$\mathcal{S} = \sum_{n'} f_{n'}(\vec{r};D) \langle \beta_n^* \beta_{n'} \rangle \quad . \tag{16}$$

Now we make use of the requirement that the random coefficients β_n and $\beta_{n'}$ be independent, as expressed by Eq. (8), which allows us to reduce the summation on n' to a single term with $n' = n$. Thus we get as a consequence of the requirement of independence of the β_n's

$$\mathcal{S} = B_n^2(D) f_n(\vec{r};D) \quad , \tag{17}$$

which, when combined with Eq. (11), gives the basic Karhunen-Loeve homogeneous integral equation

$$\int d\vec{r}\,' \, W(\vec{r}\,', D) \, C_\varphi(|\vec{r} - \vec{r}\,'|;D) \, f_n(\vec{r}\,';D) = B_n^2(D) f_n(\vec{r};D) \quad . \tag{18}$$

The eigenfunctions of this equation are the orthonormal functions we wish to work with. These functions have statistically independent coefficients in a series representative of φ as required by Eq. (8). The variance of these coefficients are the corresponding eigenvalues of the Karhunen-Loeve integral equation.

Our basic remaining tasks in this paper are 1) to see how the probability of the effective wavefront distortion, φ, being adequately small can be calculated from the eigenfunctions and eigenvalues defined by Eq. (18),

and 2) to develop an expression for C_φ to be substituted into Eq. (18). However, before delving too deeply into these matters, we first wish to consider the question of how we can "dedimensionalize" our problem so that a single case solution of the Karhunen-Loeve integral equation will provide the basic treatment for all aperture diameters and strength of turbulence conditions.

<u>Dedimensionalization</u>

The basic statistics of wavefront distortion are provided by the wave-structure function,* $\mathcal{D}(r)$, where

$$\mathcal{D}(|\vec{r} - \vec{r}\,'|) = \langle |\phi(\vec{r}) - \phi(\vec{r}\,')|^2 \rangle \quad . \tag{19}$$

It can be shown that the value of the wave-structure function may be written as

$$\mathcal{D}(r) = 6.88\,(r/r_0)^{5/3} \quad , \tag{20}$$

where r_0 is a quantity with the dimensions of length. The value of r_0 is determined by the optical wavelength in question and the distribution of the strength of turbulence along the propagation path. For our purposes here, the nature of that relationship is of no consequence. It is sufficient to know the value of r_0 as this quantity, as we shall see, completely characterizes the effective strength of turbulence for evaluation of the performance of a CENSORING system.

At this point, we state without proof the fact that we can extract the dependence of $C_\varphi(|\vec{r} - \vec{r}\,'|;D)$ on both the strength of turbulence, as

---

* It should be noted that although we have spoken of ϕ as a phase distortion, we actually consider it to be a complex phase with its real part corresponding to ordinary phase and the negative of its imaginary part corresponding to log-amplitude variations. Thus both ϕ and the random coefficients $\{\beta_n\}$ are complex quantities. In all of our analysis, we have been careful to introduce complex conjugation where appropriate, though we have not made a point of the fact that the quantities are complex. In most cases, the imaginary part is much smaller than the real part.

defined by r_0 , and the aperture diameter, D , by writing

$$C_\varphi(|\vec{r} - \vec{r}\,'|;D) = (D/r_0)^{5/3}\,\mathfrak{C}(|\vec{x} - \vec{x}\,'|) \quad , \tag{21}$$

where

$$\vec{x} = \vec{r}/D \qquad \text{and} \qquad \vec{x}\,' = \vec{r}\,'/D \quad . \tag{22}$$

We note in particular that $\mathfrak{C}(|\vec{x} - \vec{x}\,'|)$ is independent of the value of D . The validity of Eq. (21) will be established in a later section, where we shall consider in detail the evaluation of C_φ and will give an explicit expression for $\mathfrak{C}$.

If we substitute Eq. (21) into Eq. (18) and change the variables from $\vec{r},\vec{r}\,'$ to $\vec{x},\vec{x}\,'$, noting that $d\vec{r}\,'$ goes into $D^2 d\vec{x}\,'$, we get the result that

$$\int d\vec{x}\,'\; W(\vec{x}\,',1)\,\mathfrak{C}(|\vec{x} - \vec{x}\,'|)\,\mathfrak{F}_n(\vec{x}\,';D) = \mathfrak{B}_n^{\,2}(D)\,\mathfrak{F}_n(\vec{x};D) \quad , \tag{23}$$

where

$$\mathfrak{F}_n(\vec{x};D) \equiv f_n(D\vec{x};D) \quad , \tag{24}$$

and

$$\begin{aligned} \mathfrak{B}_n^{\,2}(D) &= D^{-2}\,(D/r_0)^{-5/3} B_n^{\,2}(D) \\ &= D^{-11/3}\, r_0^{\,5/3}\, B_n^{\,2}(D) \end{aligned} \tag{25}$$

We note that Eq. (23) is a homogeneous integral equation with eigenfunction $\mathfrak{F}_n(\vec{x};D)$ and eigenvalue $\mathfrak{B}_n^{\,2}(D)$. However, since the kernel of that integral equation, as well as the limits on the integration, is independent of the aperture diameter, D , then it follows that the eigenfunctions and eigenvalues must be independent of D . We therefore may write the eigenfunction $\mathfrak{F}_n(\vec{x};D)$ as $\mathfrak{F}_n(\vec{x})$ without any loss of definiteness, with the understanding

that the eigenfunction we originally sought, i.e., $f_n(\vec{r};D)$ can be written as

$$f_n(\vec{r};D) = \mathfrak{F}_n(\vec{r}/D) \quad . \tag{26}$$

Since the eigenvalues of Eq. (23), i.e., $\{\mathfrak{B}_n^2(D)\}$, are independent of D (and of r_0), we can without any loss of information write them as $\{\mathfrak{B}_n^2\}$. It then follows that the eigenvalues we originally sought, namely, $B_n^2(D)$, can be written as

$$B_n^2(D) = D^{11/3}\, r_0^{-5/3}\, \mathfrak{B}_n^2 \quad . \tag{27}$$

We recall that in accordance with the above discussion, $\{\mathfrak{B}_n^2\}$ and $\{\mathfrak{F}_n(\vec{x})\}$ are the set of eigenvalues and eigenfunctions for the Karhunen-Loeve homogeneous integral equation

$$\int d\vec{x}'\; W(\vec{x}',1)\, \mathfrak{C}(|\vec{x}-\vec{x}'|)\, \mathfrak{F}_n(\vec{x}') = \mathfrak{B}_n^2\, \mathfrak{F}_n(\vec{x}) \quad . \tag{28}$$

At this point, we need only develop an expression for $\mathfrak{C}(|\vec{x}-\vec{x}'|)$, in accordance with Eq. (21), to set up the problem for solution of the integral equation in its general form. Then, after obtaining these generalized solutions, for any value of aperture diameter, D, and turbulence parameter, r_0, we can obtain the eigenvalues and eigenfunctions, $\{B_n^2(D)\}$ and $\{f_n(\vec{r};D)\}$ from Eq.'s (26) and (27), for that particular problem. In the next section, we take up the evaluation of $\mathfrak{C}$.

Evaluation of the Kernel

Though we are interested in developing the kernel, $\mathfrak{C}(|\vec{x}-\vec{x}'|)$, we shall proceed in that by means of Eq. (21), and in most of this section shall be concerned with the evaluation of the original kernel, $C_\varphi(|\vec{r}-\vec{r}'|;D)$, as defined by Eq. (10). We shall find, at the end of this section, that extraction of a result for $\mathfrak{C}(|\vec{x}-\vec{x}'|)$ then drops out as a trivial additional manipulation.

If we substitute Eq. (4) into Eq. (10), we obtain

$$
\begin{aligned}
C_\varphi(|\vec{r} - \vec{r}\,'|;D) &= \langle [\phi(\vec{r}) - \bar{\phi} - \vec{\alpha}\cdot\vec{r}]^*[\phi(\vec{r}\,') - \bar{\phi} - \vec{\alpha}\cdot\vec{r}\,']\rangle \\
&= \langle \phi^*(\vec{r})\,\phi(\vec{r}\,')\rangle - \langle \phi^*(\vec{r})\,\bar{\phi}\rangle - \langle \phi(\vec{r}\,')\,\bar{\phi}^*\rangle + \langle \bar{\phi}^*\,\bar{\phi}\rangle \\
&\quad - \langle \phi^*(\vec{r})\,\vec{\alpha}\cdot\vec{r}\,'\rangle - \langle \phi(\vec{r}\,')\,\vec{\alpha}^*\cdot\vec{r}\rangle + \langle \bar{\phi}^*\,\vec{\alpha}\cdot\vec{r}\,'\rangle + \langle \bar{\phi}\,\vec{\alpha}^*\cdot\vec{r}\rangle \\
&\quad + \langle \vec{\alpha}^*\cdot\vec{r}\,\vec{\alpha}\cdot\vec{r}\,'\rangle \quad . \qquad (29)
\end{aligned}
$$

Making use of Eq.'s (1) and (2), and interchanging the order of ensemble averaging and integration, we can rewrite Eq. (29) in the form

$$
\begin{aligned}
C_\varphi(|\vec{r} - \vec{r}\,'|;D) &= \langle \phi^*(\vec{r})\,\phi(\vec{r}\,')\rangle \\
&\quad - (\tfrac{1}{4}\pi D^2)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'', D)\,\langle \phi^*(\vec{r})\,\phi(\vec{r}\,'')\rangle \\
&\quad - (\tfrac{1}{4}\pi D^2)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'', D)\,\langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,')\rangle \\
&\quad + (\tfrac{1}{4}\pi D^2)^{-2}\iint d\vec{r}\,''\, d\vec{r}\,'''\; W(\vec{r}\,'', D)\, W(\vec{r}\,''', D) \\
&\qquad \times \langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,''')\rangle \\
&\quad - (\tfrac{1}{64}\pi D^4)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'', D)\langle \phi^*(\vec{r})\,\phi(\vec{r}\,'')\rangle\, \vec{r}\,''\cdot\vec{r}\,' \\
&\quad - (\tfrac{1}{64}\pi D^4)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'', D)\langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,')\rangle\, \vec{r}\,''\cdot\vec{r} \\
&\quad + (\tfrac{1}{16}\pi D^3)^{-2}\iint d\vec{r}\,''\, d\vec{r}\,'''\; W(\vec{r}\,'', D)\, W(\vec{r}\,''', D) \\
&\qquad \times \langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,''')\rangle\, \vec{r}\,'\cdot\vec{r}\,''' \\
&\quad + (\tfrac{1}{16}\pi D^3)^{-2}\iint d\vec{r}\,''\, d\vec{r}\,'''\; W(\vec{r}\,'', D)\, W(\vec{r}\,''', D) \\
&\qquad \times \langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,''')\rangle\, \vec{r}\cdot\vec{r}\,'' \\
&\quad + (\tfrac{1}{64}\pi D^4)^{-2}\iint d\vec{r}\,''\, d\vec{r}\,'''\; W(\vec{r}\,'', D)\, W(\vec{r}\,''', D) \\
&\qquad \times \langle \phi^*(\vec{r}\,'')\,\phi(\vec{r}\,''')\rangle(\vec{r}\cdot\vec{r}\,'')\,(\vec{r}\,'\cdot\vec{r}\,''') \quad . \qquad (30)
\end{aligned}
$$

We note that in the first four terms on the right-hand-side of Eq. (30), if the ensemble averages are each replaced by a constant, then the sum of

the four terms vanishes. This means that we may add or subtract a constant value from the ensemble averages in these terms. We further note that if the ensemble average in any of the remaining terms is replaced by a constant, the integration is such that the term will vanish. Hence we may add or subtract a constant from any of these ensemble averages. Because of the stationarity of $\langle\phi^*(\vec{r})\,\phi(\vec{r})\rangle$, it is identical in value to $\langle\phi^*(\vec{r}\,')\,\phi(\vec{r}\,')\rangle$ or $\langle\phi^*(\vec{r}\,'')\,\phi(\vec{r}\,'')\rangle$, etc. We find then that by appropriate manipulation (which, amongst other things, includes taking note of the fact that for physical reasons C_φ is real, so that we can replace the right-hand-side of Eq. (30) with one-half the sum of its value plus its complex conjugate), we can replace each ensemble average of the product of phases on the right-hand-side of Eq. (30) by minus one-half the ensemble average of the difference of the phases absolute value square. Thus

$$\langle\phi^*(\vec{r})\,\phi(\vec{r}\,')\rangle \Rightarrow -\tfrac{1}{2}\mathcal{D}(|\vec{r}-\vec{r}\,'|) \quad , \tag{31}$$

or

$$\langle\phi^*(\vec{r}\,'')\,\phi(\vec{r}\,''')\rangle \Rightarrow -\tfrac{1}{2}\mathcal{D}(|\vec{r}\,''-\vec{r}\,'''|) \quad , \tag{32}$$

etc., where $\mathcal{D}$, the wave-structure function, is defined in Eq. (19), with the value given in Eq. (20).

We can now rewrite Eq. (30) in the form

$$C_\varphi(|\vec{r}-\vec{r}\,'|;D) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_9 \quad , \tag{33}$$

where

$$T_1 = -\tfrac{1}{2}\mathcal{D}(|\vec{r}-\vec{r}\,'|) \quad , \tag{34}$$

$$T_2 = \tfrac{1}{2}(\tfrac{1}{4}\pi D^2)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'',D)\,\mathcal{D}(|\vec{r}-\vec{r}\,''|) \quad , \tag{35}$$

$$T_3 = \tfrac{1}{2}(\tfrac{1}{4}\pi D^2)^{-1}\int d\vec{r}\,''\; W(\vec{r}\,'',D)\,\mathcal{D}(|\vec{r}\,'-\vec{r}\,''|) \quad , \tag{36}$$

$$T_4 = -\tfrac{1}{2}(\tfrac{1}{4}\pi D^2)^{-2}\iint d\vec{r}\,''\, d\vec{r}\,'''\; W(\vec{r}\,'',D)\,W(\vec{r}\,''',D) \times \mathcal{D}(|\vec{r}\,''-\vec{r}\,'''|) \quad , \tag{37}$$

$$T_5 = \tfrac{1}{2}\left(\tfrac{1}{64}\pi D^4\right)^{-1}\int d\vec{r}''\; W(\vec{r}'', D)\,\mathcal{B}(|\vec{r} - \vec{r}''|)\,\vec{r}''\cdot\vec{r}' \quad , \tag{38}$$

$$T_6 = \tfrac{1}{2}\left(\tfrac{1}{64}\pi D^4\right)^{-1}\int d\vec{r}''\; W(\vec{r}'', D)\,\mathcal{B}(|\vec{r}' - r''|)\,\vec{r}''\cdot\vec{r} \quad , \tag{39}$$

$$T_9 = -\tfrac{1}{2}\left(\tfrac{1}{64}\pi D^4\right)^{-2}\iint d\vec{r}''\, d\vec{r}'''\; W(\vec{r}'', D)\, W(\vec{r}''', D)$$

$$\times\ \mathcal{B}(|\vec{r}'' - \vec{r}'''|)\,(\vec{r}\cdot\vec{r}'')\,(\vec{r}'\cdot\vec{r}''') \quad . \tag{40}$$

In developing Eq. (33) from Eq. (29), we have dropped the two terms

$$T_7 = -\tfrac{1}{2}\left(\tfrac{1}{16}\pi D^3\right)^{-2}\iint d\vec{r}''\, d\vec{r}'''\; W(\vec{r}'', D)\, W(\vec{r}''', D)$$

$$\times\ \mathcal{B}(|\vec{r}'' - \vec{r}'''|)\,\vec{r}'\cdot\vec{r}''' \quad , \tag{41}$$

$$T_8 = -\tfrac{1}{2}\left(\tfrac{1}{16}\pi D^3\right)^{-2}\iint d\vec{r}''\, d\vec{r}'''\; W(\vec{r}'', D)\, W(\vec{r}''', D)$$

$$\times\ \mathcal{B}(\vec{r}'' - \vec{r}'''|)\,\vec{r}\cdot\vec{r}''' \quad , \tag{42}$$

which correspond to the seventh and eighth terms in Eq. (29). Our reason for doing this is that both terms can be shown to have zero value. To see that these terms do indeed vanish, we note that if we consider pairs of values of $\vec{r}''$ and $\vec{r}'''$ that hold $\vec{r}''$, $\vec{r}'''$, and the angle between $\vec{r}''$ and $\vec{r}'''$ constant, then as we integrate about the orientation of the $\vec{r}'''$-vector, everything in the integrand is constant except $\vec{r}\cdot\vec{r}'''$. As a result of the variation of this factor, the integration over 2π yields a zero value. (It is interesting to note that T_7 and T_8 correspond to the correlation of $\vec{\phi}$ and $\vec{\alpha}$, which, as we should have expected, are uncorrelated.)

With Eq. (33) established, we now look into the problem of using Eq. (20) for $\mathcal{B}$ to simplify our expression for C_Φ . It is convenient at this point to introduce the following functions:

$$\mathcal{G}_0(x) = 3.44\, x^{5/3} \quad , \tag{43}$$

$$\mathcal{G}_1(x) = 3.44\left(\tfrac{1}{4}\pi\right)^{-1}\int_0^{1/2} dx''\, x''\int_0^{2\pi} d\theta''\,(x^2 + x''^2 - 2xx''\cos\theta'')^{5/6} \tag{44}$$

$$\mathcal{G}_2 = 8\int_0^{1/2} dx''\, x''\;\mathcal{G}_1(x'') \quad , \tag{45}$$

$$\mathcal{G}_3(x) = 3.44\left(\tfrac{1}{64}\pi\right)^{-1} \int_0^{1/2} dx'' \, x''^2 \int_0^{2\pi} d\theta'' \cos\theta''$$

$$\times \left(x^2 + x''^2 - 2xx'' \cos\theta''\right)^{5/6} \quad , \qquad (46)$$

$$\mathcal{G}_4 = 64 \int_0^{1/2} dx'' \, x''^2 \, \mathcal{G}_3(x'') \quad . \qquad (47)$$

We shall see shortly that C_φ can be expressed in terms of these five functions (or more precisely, three functions and two constants).

From Eq. (20), we see that

$$T_1 = -3.44\left(r^2 + r'^2 - 2rr' \cos\theta'\right)^{5/6} r_0^{-5/3}$$

$$= -3.44\,(D/r_0)^{5/3}\left[(r/D)^2 + (r'/D)^2 - 2(r/D)(r'/D)\cos\theta'\right]^{5/6}$$

$$= -(D/r_0)^{5/3}\,\mathcal{G}_0\left(\left|(\vec{r}/D) - (\vec{r}\,'/D)\right|\right) \quad . \qquad (48)$$

Proceeding in the same vein, we see that

$$T_2 = 3.44\left(\tfrac{1}{4}\pi\right)^{-1}(D/r_0)^{5/3} \int_0^{1/2} dx'' \, x'' \int_0^{2\pi} d\theta''$$

$$\times \left[(r/D)^2 + x''^2 - 2(r/D)\,x'' \cos\theta''\right]^{5/6}$$

$$= (D/r_0)^{5/3}\,\mathcal{G}_1(r/D) \quad , \qquad (49)$$

where we have made the transformation of variables

$$\vec{r}\,'' = D\,\vec{x}\,'' \qquad \text{and} \qquad d\vec{r}\,'' = D^2\,d\vec{x}\,'' \quad , \qquad (50)$$

and later will also use

$$\vec{r}\,''' = D\,\vec{x}\,''' \qquad \text{and} \qquad d\vec{r}\,''' = D^2\,d\vec{x}\,''' \quad . \qquad (51)$$

For T_3 , working in exactly the same way but with $\vec{r}$ replaced by $\vec{r}'$, we get

$$T_3 = (D/r_0)^{5/3} \mathcal{G}_1 (r'/D) \quad . \tag{52}$$

Working in the same manner with T_4 , as defined in Eq. (37), we get

$$T_4 = -3.44 \left(\tfrac{1}{4}\pi\right)^{-2} (D/r_0)^{5/3} \int_0^{1/2} dx'' \, x'' \int_0^{2\pi} d\theta'' \int_0^{1/2} dx''' \, x''' \int_0^{2\pi} d\theta'''$$

$$\times \left[x''^2 + x'''^2 - 2x'' \, x''' \cos(\theta''' - \theta'')\right]^{5/6}$$

$$= -3.44 \left(\tfrac{1}{4}\pi\right)^{-2} (D/r_0)^{5/3} \int_0^{1/2} dx'' \, x'' \, 2\pi \left\{ \int_0^{1/2} dx''' \, x''' \int_0^{2\pi} d\theta''' \right.$$

$$\left. \times \left[x''^2 + x'''^2 - 2x'' \, x''' \cos(\theta''')\right]^{5/6} \right\} \quad . \tag{53}$$

Here we have replaced the variable θ''' by $\theta''' + \theta''$ (treating θ'' as a constant for the θ'''-integration), and then shifted the limits of that integration from $\theta'' \leftrightarrow 2\pi + \theta''$ to $0 \leftrightarrow 2\pi$. This then allowed the θ''-integration to be performed, yielding a factor of 2π for the final result in Eq. (53). We note that the quantity in the curly brackets in Eq. (53) is directly related to $\mathcal{G}_1(x'')$. Thus we can write

$$T_4 = -8 \, (D/r_0)^{5/3} \int_0^{1/2} dx'' \, x'' \, \mathcal{G}_1(x'')$$

$$= -(D/r_0)^{5/3} \mathcal{G}_2 \quad . \tag{54}$$

For T_5 as defined by Eq. (38), we have the necessary extra powers of D^{-1} in front of the integral to convert $\vec{r}'' \cdot \vec{r}'$ in the integrand into $\vec{x}'' \cdot \vec{x}'$. Following the same procedure as above, this allows us to

write

$$T_5 = 3.44\left(\tfrac{1}{64}\pi\right)^{-1}(D/r_0)^{5/3}\int_0^{1/2} dx''\, x'' \int_0^{2\pi} d\theta''\, x''\, x'\cos(\theta''-\theta')$$

$$\times\left[(r/D)^2 + x''^2 - 2x\, x''\cos\theta''\right]^{5/6} \qquad , \qquad (55)$$

where the θ'' value is referenced to the $\vec{r}$ orientation so that $\theta''=0$ when $\vec{x}''$ is parallel to $\vec{r}$ (or rather when $\vec{r}''$ is parallel to $\vec{r}$). θ' is similarly referenced to the $\vec{r}$ vector. At this point, we note that since we can rewrite $\cos(\theta''-\theta')$ in the form

$$\cos(\theta''-\theta') = \cos\theta''\cos\theta' + \sin\theta''\sin\theta' \qquad , \qquad (56)$$

then we can separate the θ''-integration in Eq. (55) into the sum of two integrations. The first would be multiplied by a factor of $\cos\theta'$ and would have $\cos(\theta''-\theta')$ in the integrand of Eq. (55) replaced by $\cos\theta''$. The second would be multiplied by $\sin\theta'$ and would have $\cos(\theta''-\theta')$ in the integrand replaced by $\sin\theta''$. We note, however, that since the integrand in the second integration would be odd in θ'' , the second integral being over the range $0 \leftrightarrow 2\pi$ will have zero value. Thus we obtain the result that

$$T_5 = 3.44\left(\tfrac{1}{64}\pi\right)^{-1}(D/r_0)^{5/3}\, x'\cos\theta' \int_0^{1/2} dx''\, x''^2 \int_0^{2\pi} d\theta''\cos\theta''$$

$$\times\left[(r/D)^2 + x''^2 - 2x\, x''\cos\theta''\right]^{5/6}$$

$$= (D/r_0)^{5/3}(r'/D)\cos\theta'\, \mathcal{G}_3\,(r/D) \quad . \qquad (57)$$

Exactly the same procedure applied to the evaluation of T_6 leads to the result

$$T_6 = (D/r_0)^{5/3}(r/D)\cos\theta'\, \mathcal{G}_3\,(r'/D) \quad . \qquad (58)$$

The evaluation of T_9 follows the same general approach as the above. In this case, we have enough extra powers of D^{-1} in front of the integral to allow us to convert $(\vec{r} \cdot \vec{r}'')(\vec{r}' \cdot \vec{r}''')$ into $[(\vec{r}/D) \cdot \vec{x}''][(\vec{r}'/D) \cdot \vec{x}''']$. Thus we can write

$$T_9 = -3.44 \left(\tfrac{1}{64}\pi\right)^{-2} \int_0^{1/2} dx''\, x'' \int_0^{2\pi} d\theta'' \int_0^{1/2} dx'''\, x''' \int_0^{2\pi} d\theta''' \,(r/D)(r'/D)\, x''\, x'''$$

$$\times \cos\theta'' \cos(\theta''' - \theta')\,[x''^2 + x'''^2 - 2x''\, x''' \cos(\theta''' - \theta'')]^{5/6}$$

$$= -3.44 \left(\tfrac{1}{64}\pi\right)^{-2} (r/D)(r'/D) \int_0^{1/2} dx''\, x''^2 \int_0^{2\pi} d\theta'' \int_0^{1/2} dx'''\, x'''^2 \int_0^{2\pi} d\theta''' \cos\theta''$$

$$\times \cos[\theta''' + (\theta'' - \theta')](x''^2 + x'''^2 - 2x''\, x''' \cos\theta''')^{5/6} \quad . \qquad (59)$$

In order to obtain the final form of Eq. (59), we have replaced θ''' by $\theta''' + \theta''$ (treating θ'' as a constant for the θ'''-integration), and then adjusted the limits of the new θ'''-integration from $\theta'' \leftrightarrow 2\pi + \theta''$ to $0 \leftrightarrow 2\pi$. Now we can rewrite $\cos[\theta''' + (\theta'' - \theta')]$ as

$$\cos[\theta''' + (\theta'' - \theta')] = \cos\theta''' \cos(\theta'' - \theta') + \sin\theta''' \sin(\theta'' - \theta') \quad , \qquad (60)$$

and using the same arguments as were used to develop Eq. (57) from Eq. (55), i.e., dropping the $\sin\theta'''$ dependence because it leads to an odd integrand in θ''', we obtain

$$T_9 = -3.44 \left(\tfrac{1}{64}\pi\right)^{-2} (r/D)(r'/D) \int_0^{1/2} dx''\, x''^2 \int_0^{2\pi} d\theta'' \cos\theta'' \cos(\theta'' - \theta')$$

$$\times \left\{ \int_0^{1/2} dx'''\, x'''^2 \int_0^{2\pi} d\theta''' \cos\theta''' (x''^2 + x'''^2 - 2x''\, x''' \cos\theta''')^{5/6} \right\} . \quad (61)$$

First of all, we note the close relationship between the quantity in the curly brackets in Eq. (61) to the integration defining $\mathcal{G}_3$ in Eq. (46). Second, we

note that the only θ''-dependence in Eq. (61) is $\cos\theta'' \cos(\theta''-\theta')$. Since

$$\int_0^{2\pi} d\theta'' \cos\theta'' \cos(\theta''-\theta') = \pi \cos\theta' \quad , \tag{62}$$

we see that Eq. (61) can be rewritten as

$$T_9 = -(D/r_0)^{5/3} (r/D)(r'/D) \cos\theta' \, \mathcal{G}_4 \quad . \tag{63}$$

With all of these results in hand, we can now rewrite C_φ as given by Eq. (33) in the form

$$\begin{aligned} C_\varphi(|\vec{r}-\vec{r}'|;D) &= C_\varphi(r\,,\,r',\,\theta'\,;\,D) \\ &= (D/r_0)^{5/3}\{-\mathcal{G}_0(|(\vec{r}/D)-(\vec{r}'/D)|) + \mathcal{G}_1(r/D) \\ &\quad + \mathcal{G}_1(r'/D) - \mathcal{G}_2 + (r'/D)\cos\theta' \, \mathcal{G}_3(r/D) \\ &\quad + (r/D)\cos\theta' \, \mathcal{G}_3(r'/D) - (r/D)(r'/D)\cos\theta' \, \mathcal{G}_4\} \,. \end{aligned} \tag{64}$$

In writing Eq. (64), we have taken the liberty of introducing the notation $C_\varphi(r\,,\,r',\,\theta'\,;\,D)$ to make it explicit that the dependence on $|\vec{r}-\vec{r}'|$ can just as well be considered a dependence on r, r', and θ'. We note that although θ' has been defined in terms of the $\vec{r}'$, physically, and in our future work, it can be considered as simply being the angle between $\vec{r}$ and $\vec{r}'$, if we choose.

The first thing we wish to note in considering Eq. (64) is the fact that it is indeed a function of $\vec{r}/D$ and $\vec{r}'/D$ multiplied by $(D/r_0)^{5/3}$, thus justifying the assertion that led us to write down Eq. (21). In fact, comparing Eq.'s (21) and (64), we see that the value of $\mathfrak{C}(|\vec{x}-\vec{x}'|)$ can be written as

$$\mathfrak{C}(|\vec{x}-\vec{x}'|) = \mathfrak{C}(x , x', \theta')$$

$$= - \mathfrak{G}_0 (|\vec{x}-\vec{x}'|) + \mathfrak{G}_1 (x) + \mathfrak{G}_1 (x') - \mathfrak{G}_2$$

$$+ x' \cos \theta' \mathfrak{G}_3 (x) + x \cos \theta' \mathfrak{G}_3 (x') - x\ x' \cos \theta' \mathfrak{G}_4 \ . \quad (65)$$

With this expression in hand for $\mathfrak{C}$, with the $\mathfrak{G}$-functions defined by Eq.'s (43) to (47), we can now turn our attention back to the problem of solving the integral equation for the eigenvalues $\mathfrak{A}_n^2$ and the eigenfunctions $\mathfrak{F}_n(\vec{x})$, in accordance with Eq. (28).

Integral Equation Reduction

As indicated at the start of this paper, we shall not attempt here to solve the Karhunen-Loeve homogeneous integral equation, the pertinent form of which is given by Eq. (28). Ultimately we intend to solve this equation using numerical techniques, in particular the Givens-Householder method. At this point, however, we are interested in numerical procedures that will simplify this equation. We note that the basic integral equation is two-dimensional and that as a consequence, the size of the matrix required to obtain any reasonable resolution over the aperture for our eigenfunctions will be unreasonably large. We propose to avoid this problem by introducing a separation of variables in the eigenfunction.

We postulate that the eigenfunction $\mathfrak{F}_n(\vec{x})$ can be separated into a radial dependence and an azimuthal dependence, and further postulate that the azimuthal dependence can be written in the form $\exp(i\,q\,\theta)$, where $q = 0, \pm 1, \pm 2, \ldots$ is a "quantum number" for a set of solutions corresponding to a subset of the n "quantum numbers." The radial dependence would have its own set of "quantum numbers", $p = 1, 2, 3, \ldots$, with each combination (p, q) corresponding to an element in the set that was "counted" by n. Thus we would write

$$\mathfrak{F}_n(\vec{x}) \equiv \mathfrak{R}_p^q(x) \exp(i\,q\,\theta) \quad , \quad (66)$$

where the superscript q over the radial dependence term, i.e., the function $\mathfrak{R}$, is used to indicate that for each value of q, we may expect to obtain a different set of radial dependence functions. The eigenvalue would now be written as $\mathfrak{B}^2_{p,q}$ in place of $\mathfrak{B}_n{}^2$.

To validate our hypothesis concerning the validity of Eq. (66), we shall substitute Eq. (66) for $\mathfrak{J}_n(\vec{x})$ into the Karhunen-Loeve homogeneous integral equation given by Eq. (28), and show that it leads to self-consistency in the sense that the form of $\mathfrak{J}_n(\vec{x})$ will be found to have the form given by Eq. (66). While this is not a rigorous proof of the validity of the separation of variables, our manipulations will define the key steps required to develop such a rigorous proof. We shall not concern ourselves further with the matter of a rigorous proof.

If we substitute Eq. (66) for $\mathfrak{J}_n(\vec{x})$ into Eq. (28), we obtain

$$\int d\vec{x}' \; W(\vec{x}', 1) \, \mathfrak{S}(|\vec{x}-\vec{x}'|) \, \mathfrak{J}_n(\vec{x}')$$

$$= \int_0^{1/2} dx' \; x' \int_0^{2\pi} d\theta' \; \mathfrak{S}(x, x', \theta' - \theta) \, \mathfrak{R}_p^q(x') \exp(i\, q\, \theta') \quad , \qquad (67)$$

where here we have chosen an arbitrary angular reference point so that we can define angles θ and θ' associated with $\vec{x}$ and $\vec{x}'$, respectively, rather than merely having θ' defined as the angle between $\vec{x}$ and $\vec{x}'$. Then in writing $\mathfrak{S}$, we took note of the fact that the θ'-dependence indicated in Eq. (65) was in this case a dependence on $\theta' - \theta$. Now if we replace θ' in Eq. (67) with $\theta' + \theta$ and then readjust the limits of the θ'-integration from $\theta \leftrightarrow 2\pi + \theta$ to $0 \leftrightarrow 2\pi$, we obtain

$$\int d\vec{x}' \; W(\vec{x}', 1) \, \mathfrak{S}(|\vec{x}-\vec{x}'|) \, \mathfrak{J}_n(\vec{x}')$$

$$= \exp(i\, q\, \theta) \int_0^{1/2} dx' \; \mathfrak{K}_q(x, x') \, \mathfrak{R}_p^q(x') \quad , \qquad (68)$$

where we have used $\mathfrak{K}_q$ to mean

$$\mathfrak{K}_q(x, x') = x' \int_0^{2\pi} d\theta' \; \mathfrak{C}(x, x', \theta') \exp(i q \theta') \quad . \tag{69}$$

Now combining Eq.'s (68) and (28), we see that

$$\mathfrak{J}_n(\vec{x}) = \exp(i q \theta) \left\{ \mathfrak{B}_n^{-2} \int_0^{1/2} dx' \, \mathfrak{K}_q(x, x') \mathfrak{R}_p^q(x') \right\} . \tag{70}$$

Comparison of Eq. (70) with Eq. (66) shows that our assumption of Eq. (66) for $\mathfrak{J}_n(\vec{x}')$ leads to self-consistent results for $\mathfrak{J}_n(\vec{x})$. Combining Eq.'s (66) and (70), we obtain the equation

$$\int_0^{1/2} dx' \, \mathfrak{K}_q(x, x') \mathfrak{R}_p^q(x') = \mathfrak{B}_{p,q}^2 \, \mathfrak{R}_p^q(x) \quad . \tag{71}$$

This is a Karhunen-Loeve integral equation type definition for the radial function, $\mathfrak{R}_p^q$, with a kernel, $\mathfrak{K}_q$, which can be different for each value of the "quantum number," q .

Eq. (71) provides us with a basis for calculation of the complete set of eigenvalues, $\mathfrak{B}_{p,q}^2$ (or $\mathfrak{B}_n^2$ in our original notation), and together with Eq. (66) provides a definition of our two-dimensional eigenfunction, $\mathfrak{R}_p^q(x) \exp(i q \theta)$ (or $\mathfrak{J}_n(\vec{x})$ in our original notation). The set of kernels, $\{\mathfrak{K}_q\}$, can be obtained by substituting Eq. (65) into Eq. (69). We get

$$\begin{aligned} \mathfrak{K}_0(x, x') = {} & -x' \int_0^{2\pi} d\theta' \, \mathfrak{G}_0([x^2 + x'^2 - 2x x' \cos \theta']^{1/2} \\ & + 2\pi x' [\mathfrak{G}_1(x) + \mathfrak{G}_1(x') - \mathfrak{G}_2] \quad , \end{aligned} \tag{72}$$

$$\begin{aligned} \mathfrak{K}_{\pm 1}(x, x') = {} & -x' \int_0^{2\pi} d\theta' \, \mathfrak{G}_0([x^2 + x'^2 - 2x x' \cos \theta']^{1/2}) \cos(\theta') \\ & + \pi x' [x' \mathfrak{G}_3(x) + x \mathfrak{G}_3(x') - x x' \mathfrak{G}_4] \quad , \end{aligned} \tag{73}$$

and for the magnitude of q greater than one

$$\mathcal{R}_q(x, x') = -x' \int_0^{2\pi} d\theta' \, \mathcal{B}_0 \left([x^2 + x'^2 - 2x\, x' \cos\theta']^{1/2}\right) \cos(q\,\theta')$$

$$\text{for } q = \pm 2\,,\ \pm 3\,,\ \pm 4\,,\ \ldots \tag{74}$$

In obtaining these results, we have made use of the fact that $\exp(i\,q\,\theta')$ can be written as $\cos\theta' + i\sin\theta'$, and noted that the $\sin\theta'$ leads to integrands odd in θ' so that their value after integration of $0 \leftrightarrow 2\pi$ vanishes.

The combination of Eq.'s (26), (27), (43), (44), (45), (46), (47), (66), (71), (72), (73), and (74) provides the basis for calculating the eigenvalues and eigenfunctions we shall need to determine the probability of an accidental occurrence of a low energy wavefront distortion condition so that the CENSORING system will be able to produce a near diffraction-limited image. In the next section, we take up the problem of calculating this probability, given the set of eigenvalues. As noted before, we leave the problem of numerically evaluating the eigenvalues and eigenfunctions for treatment in a subsequent paper.

Probability Formulation

The key to the evaluation of the probabilities associated with a CENSORING system's performance is to recognize that this is essentially equivalent to a study of the probability distribution of the effective mean-square wavefront distortion over the aperture. The term "effective" as used here refers to the fact that we are only interested in wavefront variations excluding tilt and average phase variations -- i.e., the effective wavefront distortion is to be calculated from $\varphi(\vec{r};D)$ and not from $\phi(\vec{r})$. We write the mean-square wavefront distortion over the aperture as

$$\Delta^2 = \left(\tfrac{1}{4}\pi D^2\right)^{-1} \int d\vec{r}\; W(\vec{r};D)\, |\varphi(\vec{r};D)|^2 \quad . \tag{75}$$

It should be recognized that the term "mean" in reference to Δ^2 refers to an average over the aperture and not to an ensemble average. Thus just as the effective wavefront distortion, φ , is a random function, we see from Eq. (75) that Δ^2 is a random variable. If, at some instant, Δ^2 is small enough, then we may expect nearly diffraction-limited quality for an image formed at that instant. The problem we face in calculating CENSORING system performance is one of calculating the probability that Δ^2 will be small enough. We set as a nominal threshold the requirement that $\Delta^2 < \Delta_T^2$ radian-square as the dividing line between good and poor images. Our problem is to calculate the probability of Δ^2 being less than Δ_T^2 , this being the probability that the CENSORING system will see good enough conditions to allow an image to be formed.

If we substitute Eq. (6) into Eq. (75), we get

$$\Delta^2 = (\tfrac{1}{4}\pi D^2)^{-1} \sum_{n,n'} \beta_n^* \beta_{n'} \int d\vec{r}\, W(\vec{r}, D)\, f_n^*(\vec{r})\, f_{n'}(\vec{r}) \quad . \tag{76}$$

Now making use of the orthonormality of f_n , as defined in Eq. (5), we can reduce Eq. (76) to the form

$$\begin{aligned} \Delta^2 &= (\tfrac{1}{4}\pi D^2)^{-1} \sum_n \beta_n^* \beta_n \\ &= \sum_n [\beta_n/(\tfrac{1}{4}\pi D^2)^{1/2}]^* [\beta_n/(\tfrac{1}{4}\pi D^2)] \quad . \end{aligned} \tag{77}$$

Thus the mean-square wavefront distortion is seen to be the sum of the square of a set of gaussian random variables, $\beta_n/(\frac{1}{4}\pi D^2)^{1/2}$. We recall that according to Eq. (8), the random variable β_n has a variance given by the eigenvalue $B_n^2(D)$, so that the variance of $\beta_n/(\frac{1}{4}\pi D^2)^{1/2}$ can be written, using Eq. (27), in the form

$$\mathrm{Var}\,[\beta_n/(\tfrac{1}{4}\pi D^2)^{1/2}] = (\tfrac{1}{4}\pi D^2)^{-1} B_n^2(D)$$

$$= (D/r_0)^{5/3} (4/\pi) \mathfrak{B}_n^2 \quad . \qquad (78)$$

For convenience, we shall denote this variance by σ_n^2 (or $\sigma_{p,q}^2$) as appropriate.

$$\sigma_n^2 = \mathrm{Var}\,[\beta_n/(\tfrac{1}{4}\pi D^2)^{1/2}] \quad . \qquad (79)$$

If we can compute the eigenvalues, $\mathfrak{B}_n^2$ (or $\mathfrak{B}_{p,q}^2$) for the dedimensionalized Karhunen-Loeve Homogeneous Integral equation of Eq. (28) [or of Eq. (71)], then we can immediately write

$$\sigma_n^2 = (D/r_0)^{5/3} (4/\pi) \mathfrak{B}_n^2 \quad , \qquad (80)$$

or

$$\sigma_{p,q}^2 = (D/r_0)^{5/3} (4/\pi) \mathfrak{B}_{p,q}^2 \quad . \qquad (80')$$

It now follows that the probability of CENSORING system at any instant seeing low enough distortion to allow an image to be formed can be written as

$$P_{Censor} = \text{Prob (CENSORING System Forming an Image)}$$

$$= \text{Prob}\,(\Delta^2 \leq \Delta_T^2) \quad . \qquad (81)$$

Since the random variables β_n are independent and gaussian distributed with variance σ_n^2 , it follows that

$$P_{Censor} = \prod_{n=1}^{\infty} (2\pi\,\sigma_n^2)^{-1/2} \int_{Limits} dx_n \, \exp\,(-\tfrac{1}{2}\, x_n^2/\sigma_n^2) \quad , \qquad (82)$$

where the limits on the integration are to be understood as a composite limit on the product, or rather on the n-tuple multiple integral. The limit

corresponds to

$$\text{Limit} = \left(\sum_{n=1}^{\infty} x_n^2 \le \Delta_T^2 \right) \quad . \tag{83}$$

Eq.'s (82) and (83), in concept at least, provide a basis for the calculation of P_{Censor} , i.e., the probability of a CENSORING system image being formed. However, because of the infinite limits on n in these two equations, no practical calculations can be performed.

To provide a practical basis for carrying out the calculation of P_{Censor} , we need to truncate the series. To do this, we first note that in accordance with results which we have previously obtained elsewhere,[5] we know that the ensemble average value of Δ^2 can be written as

$$\langle \Delta^2 \rangle = 0.1345 \, (D/r_0)^{5/3} \quad . \tag{84}$$

Now if we assume that the eigenfunctions are arranged in such an order that σ_n^2 is monotonically decreasing with n , then we know that if N is large enough so that

$$\sum_{n=1}^{N} \sigma_n^2 = \langle \Delta^2 \rangle - \epsilon \, \Delta_{Tn} \quad , \tag{85}$$

then if ϵ has been chosen to be a small enough quantity, we may consider N , rather than ∞ , to be the practical upper limit on the n dependencies in Eq.'s (82) and (83). As a practical matter, we would replace Δ_T^2 in Eq. (83) with $\Delta_T^2(1-\epsilon)$.

This has the effect of saying that above some value of n (namely n = N), we are not particularly concerned with the exact amount of wavefront distortion introduced by each degree of freedom. The exact value

of those β_n's does not concern us since we know that the corresponding variances, σ_n^2 , are so small that the values of the β_n's will be tolerably small. We expect the contribution of all those higher order terms, which we are suppressing, to the mean-square wavefront distortion to only be of the order of $\epsilon\Delta_T^2$, which, by our choice of ϵ , we have made sure is tolerably small.

In an actual calculation of P_{Censor} , we would first compute an ordered series of eigenvalues $\mathfrak{B}_n^2$. Then using Eq. (80), we would compute the variances, σ_n^2 . By applying Eq. (85), we could then determine the truncation level, N . At that point, our calculation would reduce to the evaluation of the integral

$$P_{Censor} = \underset{Limit}{\iint} \cdots \int dx_1\, dx_2 \ldots dx_N \prod_{n=1}^{N} (2\pi\, \sigma_n^2)^{-1/2} \exp\left(-\tfrac{1}{2}\, x_n^2/\sigma_n^2\right), \quad (86)$$

where

$$\text{Limit} \equiv \left(\sum_{n=1}^{N} x_n^2 \leq \Delta_T^2 (1-\epsilon) \right) \quad . \qquad (87)$$

Our problem is thus reduced to the numerical evaluation of the integral in Eq. (86), having first solved the Karhunen-Loeve homogeneous integral equation for the eigenvalues. Neither is a trivial numerical task, but in practice can be expected to be rather straightforward, if somewhat ponderous in terms of required computer effort. These tasks are taken up in Part II of this report.

PART II

Numerical Evaluation of Probabilities Governing the Performance of a CENSORING System

Introduction

In Part I of this report, a formal basis was developed for the analysis of the expected performance of a CENSORING system. The basic quantity of interest was identified as the probability that at any instant of time, the wavefront distortion over the entrance aperture would have an rms deviation from a plane of less than one radian. (The term "rms" is used here in the sense of an average over the aperture.) Because the CENSORING system is designed to form a short exposure image, the wavefront deviation is to be measured relative to the optimally chosen tilted plane, i.e., that plane whose tilt is such as to minimize the rms deviation.

It was shown that the probability of interest could be calculated in terms of a multi-dimensional gaussian distribution in a hyper-space, where the hyper-space was defined in terms of a set of functions which could be used to decompose a sample of the randomly distorted wavefront taken over the aperture into a set of statistically independent components. Because the wavefront distortion is a gaussian random process,[1] and because the magnitude of each of these independent random components is obtained by a linear process from the random wavefront distortion, it follows that each component is a gaussian random variable. The random amplitude of each component represents one of the dimensions in the hyper-space, and the probability of interest is the probability that all of the random variables will take on small enough values at some instant of time.

Because the functions used for the decomposition of the wavefront are a set of functions that are orthonormal over the space defined by the CENSORING system's aperture, it follows that the mean square deviation of the wavefront at any instant (with the mean taken as an average over the aperture) is just equal to the sum of the square of the component amplitudes

in the wavefront decomposition. This means that the probability of the rms wavefront distortion being less than or equal to one-radian is just the probability that the random components will define a point in the hyper-space that lies within a hyper-sphere of one-radian radius and centered at the origin. Since the components each obey an independent gaussian distribution, the problem of evaluating the probability of interest can be seen to reduce to a multi-dimensional integral within a unit hyper-sphere of a set of gaussian distributions. All we need in order to be able to carry out this evaluation is information on the variances to be associated with each element of the set of gaussian distribution. In the previous work, it was shown that these variances could be obtained by solving the Karhunen-Loève integral equation associated with the wavefront distortion statistics, and that they were proportional to the eigenvalues of that equation.

The single wavefront distortion statistic required for this problem is the wave structure function, $\mathcal{D}(r)$, where

$$\mathcal{D}(r) = 6.88\,(r/r_0)^{5/3} \quad . \tag{1}$$

The orthonormal function set $\{f_n(\vec{r})\}$ and the associated set of variances, $\{\sigma_n^2\}$, which appropriately decompose the distorted wavefront and which define the gaussian probability distributions of interest in our hyper-space integration have been shown to correspond to the eigenfunctions and to be proportional to the eigenvalues of a Karhunen-Loève integral equation utilizing a function of $\mathcal{D}(r)$ as the kernel of the integral. The relationship between the set of variances of interest and the set of eigenvalues, $\{B_n^2\}$, is given by the equation

$$\sigma_n^2 = B_n^2/(\tfrac{1}{4}\pi D^2) \quad . \tag{2}$$

In seeking a solution of this integral equation, it has been shown that the set $\{f_n(\vec{r})\}$ can be decomposed in subsets by separation of variables

into polar coordinates, i. e.,

$$\vec{r} \equiv (r, \theta) \quad . \tag{3}$$

It has been shown that we can write

$$f_n(\vec{r}) = R_p^q(r) \exp(i\, q\, \theta) \;, \quad q = -\infty, \ldots -2, -1, 0, 1, 2, \ldots +\infty$$
$$p = 1, 2, 3, \ldots +\infty \quad , \tag{4}$$

where the function $R_p^q(r)$ represents a set, on p, of functions that satisfy a homogeneous integral equation with a kernel that is different for each value of q. (Actually the kernel for q and $-q$ are identical.) The eigenvalues of this integral equation, $B_{p,q}^2$ can be equated with the eigenvalues of the original integral equation, i.e.,

$$B_{p,q}^2 = B_n^2 \quad , \tag{5}$$

and equivalently, the variance of interest can be written in the form $\sigma_{p,q}^2$, where

$$\sigma_{p,q}^2 = \sigma_n^2 \quad . \tag{6}$$

The integral equation defining $R_p^q(r)$ and $B_{p,q}^2$ involves the aperture diameter of the CENSORING system, D, and the basic wavefront distortion turbulence parameter, r_0. It is convenient to cast the integral equation in a form which is independent of these two parameters so that a single set of numerical solutions to the integral equation can be applied for all possible values of D and r_0. It has been shown that if we define the eigenfunctions, $\mathfrak{R}_p^q(x)$ and $\mathfrak{B}_{p,q}^2$ by the integral equation

$$\int_0^{1/2} dx' \, \mathfrak{K}_q(x, x') = \mathfrak{B}_{p,q}^2 \, \mathfrak{R}_p^q(x) \quad , \tag{7}$$

then

$$B_n^{\,2} \;\equiv\; B_{p,q}^{\,2} \;=\; D^{11/3}\, r_0^{-5/3}\, \mathfrak{B}_{p,q}^{\,2} \qquad , \tag{8}$$

and

$$f_n(\vec{r}) = R_p^{\,q}(r) \exp(i\, q\, \theta) = \mathfrak{R}_p^{\,q}(r/D) \exp(i\, q\, \theta) \qquad . \tag{9}$$

If we define $\tilde{\mathfrak{K}}_q(x, x')$ as

$$\tilde{\mathfrak{K}}_q(x, x') = -\, x' \int_0^{2\pi} d\theta'\, \mathfrak{G}_0\left([x^2 + x'^2 - 2xx' \cos \theta']^{1/2}\right) \cos(q\theta')\ , \tag{10}$$

then the kernel for the integral equation of Eq. (7) can be written as

$$\mathfrak{K}_q(x, x') = \tilde{\mathfrak{K}}_q(x, x') \qquad \text{if} \quad q = \pm 2,\ \pm 3,\ \pm 4, \ldots , \tag{11}$$

$$\mathfrak{K}_1(x, x') = \tilde{\mathfrak{K}}_1(x, x') + \pi\, x'[x'\mathfrak{G}_3(x) + x\mathfrak{G}_3(x') - xx'\,\mathfrak{G}_4] \quad , \tag{12}$$

$$\mathfrak{K}_0(x, x') = \tilde{\mathfrak{K}}_0(x, x') + 2\pi\, x'[\mathfrak{G}_1(x) + \mathfrak{G}_1(x') - \mathfrak{G}_2] \quad . \tag{13}$$

The $\mathfrak{G}$-functions are defined as

$$\mathfrak{G}_0(x) \;=\; 3.44\, x^{5/3} \qquad , \tag{14}$$

$$\mathfrak{G}_1(x) = 3.44\, (\tfrac{1}{4}\pi)^{-1} \int_0^{1/2} dx''\, x'' \int_0^{2\pi} d\theta''(x^2 + x''^2 - 2xx'' \cos \theta'')^{5/6}, \tag{15}$$

$$\mathfrak{G}_2 \;=\; 8 \int_0^{1/2} dx''\, x''\ \mathfrak{G}_1(x'') \quad , \tag{16}$$

$$\mathfrak{G}_3(x) = 3.44\, (\tfrac{1}{4}\pi)^{-1} \int_0^{1/2} dx''\, x''^2 \int_0^{2\pi} d\theta''\, \cos \theta''(x^2 + x''^2 - 2xx'' \cos \theta'')^{5/6} \qquad , \tag{17}$$

$$\mathfrak{G}_4 \;=\; 64 \int_0^{1/2} dx''\, x''^2\, \mathfrak{G}_3(x'') \qquad . \tag{18}$$

We note in passing that if we had been interested in a system in which, even during a short exposure, wavefront tilt had to be considered a portion of the wavefront distortion (which it does not have to be for normal short exposure purposes), then the only change in the above results would have been to modify Eq. (12) to the form

$$\mathfrak{R}_1(x, x') = \tilde{\mathfrak{R}}_1(x, x') \quad . \tag{12'}$$

Our basic problem is to solve the integral Eq. (7) for all of the eigenvalues, $\{\mathfrak{B}_{p,q}{}^2\}$ and then for a particular set of values of D and r_0 obtain the corresponding set of variances $\{\sigma_{p,q}{}^2\}$ in accordance with Eq.'s (2) and (8), from which we get

$$\sigma_{p,q}{}^2 = \frac{4}{\pi}\left(\frac{D}{r_0}\right)^{5/3} \mathfrak{B}_{p,q}{}^2 \quad . \tag{19}$$

The second half of the problem is the evaluation of the integral in hyper-space which defines the probability that at any instant the random wave-front distortion relative to the optimally chosen tilted plane will have a mean square value, averaged over the aperture, of one radian squared or less. This probability can be written as

$$P_{CENSOR} = \prod_{p,q} (2\pi\, \sigma_{p,q}{}^2)^{-1/2} \int_{Sph} dx_{p,q} \exp\left(-\tfrac{1}{2}\, x_{p,q}{}^2/\sigma_{p,q}{}^2\right) \quad , \tag{20}$$

where the limits on the integral correspond to a hyper-sphere for which

$$\sum_{p,q} x_{p,q}{}^2 \leq 1 \quad . \tag{21}$$

In Eq.'s (20) and (21), the product and the summation run over all possible combinations of values of p and q .

In the next section, we take up the problem of casting the integral equation of Eq. (7) in a form suitable for numerical evaluation. In the sections after that, we shall first present the numerical solution technique and results, and then go into the problem of formulating an explicit form for the hyper-space integral of Eq. (20) for various values of D/r_0 . Then we shall move on to consider the evaluation of that hyper-space probability integral. Based on the results of this evaluation, we shall present a general discussion of the expected performance of a CENSORING system.

Integral Equation Numerical Formulation

The numerical solution of the Karhunen-Loève integral equation presented in Eq. (7) can be developed using standard numerical techniques. However, it will greatly simplify our numerical treatment if we first recast that equation into an equivalent form in which the kernel, i.e., $\mathcal{R}_q(x, x')$ is replaced by a kernel that manifests symmetry between x and x' . [We note that the leading factor of x' in the right hand side of Eq.'s (10), (12), and (13) destroys the symmetry of $\mathcal{R}_q(x, x')$.]

In order to obtain the desired symmetry, we introduce the following symmetrizing functions:

$$\tilde{\mathcal{R}}_q^s(x, x') = -(x x')^{1/2} \int_0^{2\pi} d\theta' \, \mathcal{G}_0([x^2 + x'^2 - 2xx' \cos\theta']^{1/2}) \cos(q\theta') , \quad (22)$$

$$\mathcal{R}_0^s(x, x') = \tilde{\mathcal{R}}_0^s(x, x') + 2\pi (x\ x')^{1/2} [\mathcal{G}_1(x) + \mathcal{G}_1(x') - \mathcal{G}_2] , \quad (23)$$

$$\mathcal{R}_1^s(x, x') = \tilde{\mathcal{R}}_1^s(x, x') + \pi (x x')^{1/2} [\mathcal{G}_3(x) + \mathcal{G}_3(x') - \mathcal{G}_4] , \quad (24)$$

$$\mathcal{R}_{-1}^s(x, x') = \mathcal{R}_1^s(x, x') , \quad (25)$$

$${}^s\mathfrak{R}_p{}^q(x) = x^{1/2} \mathfrak{R}_p{}^q(x) . \quad (26)$$

Now it follows from direct substitution that Eq. (7) can be rewritten as

$$\int_0^{1/2} dx' \, \mathfrak{K}_q{}^s(x,x') \, {}^s\mathfrak{R}_p{}^q(x') = \mathfrak{B}_{p,q}{}^2 \, {}^s\mathfrak{R}_p{}^q(x) \quad , \tag{27}$$

which integral equation has a symmetric kernel and can be solved using straightforward numerical techniques. The eigenvalues of Eq. (27) are identical to those of Eq. (7), and the eigenfunctions of Eq. (7), i.e., $\mathfrak{R}_p{}^q(x)$ can be obtained from the eigenfunctions of Eq. (27), i.e., ${}^s\mathfrak{R}_p{}^q(x)$ by use of Eq. (26).

To obtain the eigenvalues and eigenfunctions of Eq. (27), it is necessary to transform the integration into a summation, thereby obtaining a homogeneous set of simultaneous equations with a determinant that determines the eigenvalues and eigenfunctions. In order to replace the integration by a summation, we subdivide the range $x = 0$ to 0.5 into 20 sections and consider the values of x at the midpoint of each section, which we denote by x_i for $i = 1$ to 20. We can then make the replacement

$$\int_0^{1/2} dx' \, \mathfrak{K}_q{}^s(x,x') \, {}^s\mathfrak{R}_p{}^q(x') \Rightarrow \sum_{i=1}^{20} K_q{}^s(i,i') \, {}^sR_p{}^q(i') \quad , \tag{28}$$

which allows us to rewrite Eq. (27) as

$$\sum_{i'=1}^{20} K_q{}^s(i,i') \, {}^sR_p{}^q(i') = \mathfrak{B}_{p,q}{}^2 \, {}^sR_p{}^q(i) \quad , \tag{29}$$

where

$$R_p{}^q(i) \equiv {}^s\mathfrak{R}_p{}^q(x_i) \quad , \tag{30}$$

and

$$K_q{}^s(i,i') = (\tfrac{1}{2}/20) \, \mathfrak{K}_q{}^s(x_i, x_{i'}) \quad . \tag{31}$$

Eq. (29) presents us with the straightforward problem of obtaining the eigenvalues and eigenfunctions of the matrix $K_q{}^s(i,i')$. This is a

standard problem in numerical analysis. The 20 x 20 size of the matrix means that we are dealing with a rather small problem as such computations go on a large high-speed digital computer. The mathematical procedure that we shall use is based on the well-known Givens-Householder algorithm[2] and associated procedures. (The details of the actual computation will not be discussed, as the computer program utilized a proprietary CDC subroutine for this task.)

Computationally, the evaluation of $K_q^s(i, i')$ matrix elements represented almost as large a task as the determination of the eigenvalues and eigenfunctions of this matrix. The pertinent expressions can be written as

$$\begin{aligned} K_0^s(i, i') = 6.88\,(.025)(x_i\, x_{i'})^{1/2} \Big\{ & - \int_0^{\pi} d\theta\, (x_i^2 + x_{i'}^2 - 2x_i\, x_{i'} \cos\theta)^{5/6} \\ & + 8 \int_0^{1/2} du\, u \int_0^{\pi} d\theta \left[(x_i^2 + u^2 - 2x_i\, u \cos\theta)^{5/6} + (x_{i'}^2 + u^2 \right. \\ & \left. - 2\, x_{i'}\, u \cos\theta)^{5/6} \right] \\ & - 64 \int_0^{1/2} du\, u \int_0^{1/2} dv\, v \int_0^{\pi} d\theta\, (u^2 + v^2 - 2\, u\, v \cos\theta)^{5/6} \Big\} \,, \end{aligned} \tag{32}$$

$$\begin{aligned} K_{\pm 1}^s(i, i') = 6.88\,(.025)(x_i\, x_{i'})^{1/2} \Big\{ & - \int_0^{\pi} d\theta\, (x_i^2 + x_{i'}^2 - 2x_i\, x_{i'} \cos\theta)^{5/6} \cos\theta \\ & + 64 \int_0^{1/2} du\, u^2 \int_0^{\pi} d\theta \left[x_i\, (x_{i'}^2 + u^2 - 2\, x_{i'}\, u \cos\theta)^{5/6} + x_{i'} (x_i^2 \right. \\ & \left. + u^2 - 2\, x_i\, u \cos\theta)^{5/6} \right] \cos\theta \\ & - 4096\, x_i\, x_{i'} \int_0^{1/2} du\, u^2 \int_0^{1/2} dv\, v^2 \int_0^{\pi} d\theta\, (u^2 + v^2 - 2\, u\, v \cos\theta)^{5/6} \cos\theta \Big\} , \end{aligned} \tag{33}$$

and

$$K_q^s(i, i') = -6.88\,(.025)(x_i\, x_{i'})^{1/2} \int_0^{\pi} d\theta\, (x_i^2 + x_{i'}^2 - 2\, x_i\, x_{i'} \cos\theta)^{5/6} \cos(q\theta)\,, \quad \text{for } |q| > 1 \,. \tag{34}$$

We have utilized the trapezoidal rule to carry out the θ-integrations in Eq.'s (32), (33), and (34) and Romberg interpolation[3], and have used a 10-point Gaussian quadrature to evaluate the u- and v-integrations.

For a value of q much greater than two (2), there is a potential accuracy problem in the evaluation of the integral in Eq. (34). This is most easily studied if we split the θ-integration in Eq. (34) in such a way that each region of integration goes over only one-half cycle of the oscillating function $\cos(q\theta)$. This gives us

$$\int_0^{\pi} d\theta \, (x_1^2 + x_{1'}^2 - 2x_1 x_{1'} \cos\theta)^{5/6} \cos(q\theta)$$

$$= \sum_{k=1}^{q} \int_{(k-1)\pi/q}^{k\pi/q} d\theta \, (x_1^2 + x_{1'}^2 - 2x_1 x_{1'} \cos\theta)^{5/6} \cos(q\theta) \ . \quad (35)$$

The factor $(x_1^2 + x_{1'}^2 - 2x_1 x_{1'} \cos\theta)^{5/6}$ is a positive monotonically-increasing function of θ in Eq. (35), from which we see that the right-hand-side of Eq. (35) consists of the sum of a set of alternating sign terms. When there is a large difference between x_1 and $x_{1'}$, we note that $(x_1^2 + x_{1'}^2 - 2x_1 x_{1'} \times \cos\theta)^{5/6}$ is a very weak function of θ , so that the terms being summed are nearly equal in magnitude, but of alternating sign -- a situation that can seriously stress the accuracy of the computed results.

To avoid this accuracy problem, we can expand the integrand in Eq. (34) in the form

$$\int_0^{\pi} d\theta \, (x_1^2 + x_{1'}^2 - 2x_1 x_{1'} \cos\theta)^{5/6} \cos(q\theta) = S^{5/6} \int_0^{\pi} d\theta \, (1 - \epsilon \cos\theta)^{5/6} \cos(q\theta)$$

$$= S^{5/6} \int_0^{\pi} d\theta \left[1 - \frac{5}{6} \epsilon \cos\theta + \frac{5}{6}\left(-\frac{1}{6}\right)\frac{1}{2!} \epsilon^2 \cos^2\theta + \ldots \right] \cos(q\theta)$$

$$= S^{5/6} \sum_{n=1}^{\infty} \frac{P_n \epsilon^n}{n!} \int_0^{\pi} d\theta \cos^n\theta \cos(q\theta) \quad , \quad (36)$$

where

$$S = x_\ell^2 + x_{\ell'}^2 \quad , \tag{37}$$

$$\epsilon = 2\, x_\ell\, x_{\ell'} / S \quad , \tag{38}$$

$$P_n = \frac{5}{6}\left(\frac{5}{6} - 1\right)\left(\frac{5}{6} - 2\right) \cdots \left(\frac{5}{6} - n+1\right) \quad . \tag{39}$$

The integral in the final form of the right-hand-side of Eq. (36) can be shown to have the value[4]

$$\int_0^\pi d\theta \cos^n\theta \cos(q\theta) = \begin{cases} 0 & \text{if } n < q \\ 0 & \text{if } n+q = \text{odd} \\ \dfrac{\pi\, n!}{2^n \left(\frac{n+q}{2}\right)! \left(\frac{n-q}{2}\right)!} & \text{if } n+q = \text{even} \end{cases} \quad , \tag{40}$$

so that Eq. (36) may be rewritten as

$$\int_0^\pi d\theta\, (x_\ell^2 + x_{\ell'}^2 - 2\, x_\ell\, x_{\ell'} \cos\theta)^{5/6} \cos(q\theta)$$

$$= \pi\, S^{5/6} \sum_{m=0}^{\infty} \frac{P_{q+2m}\, (\epsilon/2)^{q+2m}}{(q+m)!\, m!} \quad . \tag{41}$$

For $\epsilon < .5$, which corresponds to x_ℓ and $x_{\ell'}$ significantly different in size, Eq. (41) is fairly rapidly convergent and a 40-term summation yields sufficient accuracy. For $\epsilon > .5$, x_ℓ and $x_{\ell'}$ are sufficiently close in value that $(x_\ell^2 + x_{\ell'}^2 - 2\, x_\ell\, x_{\ell'} \cos\theta)^{5/6}$ is a significant function of θ , and the evaluation of the integral in Eq. (34) can proceed by straightforward numerical quadrature without excessive loss of accuracy. For q greater than two (2), we have evaluated the θ-integration in Eq. (34) using either ordinary numerical integration techniques if $2\, x_\ell\, x_{\ell'} / (x_\ell^2 + x_{\ell'}^2)$ is greater than one-half, or used Eq. (41) for values less than one-half.

With these numerical techniques, the $K_q^s(i, i')$ matrices were evaluated for $q = 0$ to 41. These results are listed in Table I, giving he matrix in upper right triangular form. With these matrices, we were then able to carry out a determination of the eigenfunctions and eigenvalues. These are listed in Table II. The eigenfunctions as listed here have been restored to their unsymmetrized form, i. e., $\Re_p{}^q(x)$ instead of ${}^s\Re_p{}^q(x)$ by making use of Eq. (26).

The set of all eigenvalues were rank-ordered without regard to p- and q-values. In this procedure, we counted each eigenvalue twice if its q-value was not zero, since it then applied to both q and $-q$. This set of eigenvalues is listed in Table III. We note that the leading eigenvalue for each value of q for $q \geq 4$ appear in order according to the value of q. Since we only worked with $q \leq 41$, and since the leading eigenvalue for $q = 41$ is the 569, we can probably consider the list of eigenvalues complete up to the 569, or thereabout. The sum of all the eigenvalues listed, of which there are 1660, is 0.105127. This is in good agreement with the value of 0.1056 expected for the total of all eigenvalues, as derived from an earlier work which considered the expected mean square wavefront distortion.[6] We note that the cumulative sum at the 569th eigenvalue is $0.104708/0.10527 = 99.60\%$ of the total of the eigenvalues listed, and $0.104708/.1056 = 99.15\%$ of the total of all the eigenvalues.

With this list of eigenvalues, it is possible to proceed immediately to the imaging probability evaluation aspect of the problem. This we take up in the next section. Before turning to that, however, we first note that because of the ease with which we could adapt our mathematics to the case in which tilt is considered to be a significant wavefront distortion, we have carried out such calculations. As noted previously, this involves nothing more complex than using Eq. (34) in place of Eq. (33) for $q = 1$. With

this replacement, we obtain the symmetrized kernel shown in Table Ia, and the eigenvalues and eigenfunctions (with symmetrization removed) shown in Table IIa. It is particularly interesting to note that the first eigenvalue in this case is about $3\frac{1}{2}$ times larger than the sum of all the eigenvalues when tilt effects are suppressed as not contributing to effective wavefront distortion, and that the first eigenfunction appears to be very nearly a simple tilt. We also note that the second and subsequent eigenvalues with tilt distortion allowed (Table IIa) are very nearly equivalent to the first and subsequent eigenvalues when tilt distortion is not allowed (Table II, q = 1).

Probability Integrals

Having the list of eigenvalues given in Table III reliable out to the 569^{th} eigenvalue, and thus reliably containing more than 99% of the sum of all eigenvalues, we are now in a position to start the evaluation of the probability integral governing the performance of a CENSORING system. We recall that this integral is given by Eq. (20). Rewriting this to work with the n-notation (overall rank order) of Table III, rather than with the p,q-notation, we write

$$P_{CENSOR} = \prod_n (2\pi \sigma_n^2)^{-1/2} \int_{Sph} dx_n \exp(-\tfrac{1}{2} x_n^2/\sigma_n^2) \quad , \tag{42}$$

where the "Sph" limit on the n-dimensional integration corresponds to the constraint

$$\sum_n x_n^2 \le 1 \quad . \tag{43}$$

In accordance with Eq. (19), with $\mathfrak{A}_n$ denoting the n^{th} eigenvalue in Table III, we have

$$\sigma_n^2 = \frac{4}{\pi} (D/r_0)^{5/3} \mathfrak{A}_n^2 \quad , \tag{44}$$

for an aperture of diameter D with wavefront distortion characterized by the length r_0 .

We plan to evaluate the n-dimensional integral in Eq. (42) by Monte Carlo techniques. As an immediate reduction in the magnitude of the problem, we recognize that σ_n^2 decreases rather rapidly with increasing n , and that beyond some cut-off value of n , the expected value of the sum of the square of the random variables is very tightly constrained to the sum of the σ_n^2 , with very little variability. The real variability in the overall sum of the squares comes from the much fewer x_n's for which σ_n^2 is large and decreases rapidly with increasing n .

To establish the cut-off value of n , namely, N_c , we arbitrarily allow the random variables x_n beyond the cut-off to have an expected value of 0.1 for the sum of their values squared, i.e.,

$$\langle \sum_{N_c}^{\infty} x_n^2 \rangle = \sum_{N_c}^{\infty} \sigma_n^2 = 0.1 \quad . \tag{45}$$

Since we know that

$$\sum_{1}^{\infty} \sigma_n^2 = 0.1345\,(D/r_0)^{5/3} \quad , \tag{46}$$

then we can obtain N_c from the running cumulative value of the eigenvalues in Table III. We write in accordance with Eq. (44)

$$\frac{4}{\pi}(D/r_0)^{5/3} \sum_{1}^{N_c} \mathcal{B}_n^2 = 0.1345\,(D/r_0)^{5/3} - 0.1 \quad , \tag{47}$$

from which it follows that

$$\sum_{1}^{N_c} \mathcal{B}_n^2 = 0.1056 - 0.07854/(D/r_0)^{5/3} \quad . \tag{48}$$

We see that to avoid pushing the cut-off, N_c, beyond the reliable limit of our eigenvalue table, i.e., beyond about 569, the largest value of D/r_0 we can use is 14.60 (which we take to be 15).

For each value of D/r_0, we determine N_c in accordance with Eq. (48) and Table II, and then proceed with the evaluation of the truncated probability integral of Eq.'s (42) and (43), which we now rewrite as

$$P_{SENSOR} = \prod_{1}^{N_c} (2\pi \sigma_n^2)^{-1/2} \int_{sph} dx_n \exp(-\tfrac{1}{2} x_n^2/\sigma_n^2) \quad , \tag{49}$$

where now the limit on the N_c-dimensional integration is given by the constraint

$$\sum_{1}^{N_c} x_n^2 \leq 0.9 \quad . \tag{50}$$

In the evaluation of the integral in Eq. (49) by Monte Carlo methods, there are a number of approaches to the random sampling that can be used. First, and most obvious, we consider selecting points uniformly distributed in the hypersphere of Eq. (50), and evaluate the integrand of each point. Unfortunately, the integrand will be very small for most of the points selected, since many values of σ_n, for large n, will be much less than unity. This will give very poor sampling efficiency and an unmanageably large number of samples will be needed to yield even modest accuracy.

A second approach to the random sampling is to select the random points in accordance with the gaussian probability distributions for each dimension inherent in the integrand in Eq. (49). Then the integral would be evaluated by counting a one for each such randomly selected hyper-space

point that satisfied Eq. (50), and zero for each point that did not, and taking the average of these counts of one and zero. Unfortunately, this method also suffers from the problem that an unmanageably large number of sample points are needed to obtain acceptable accuracy in the integral evaluation. In this case, the problem is associated with the smaller values of n, especially for larger values of D/r_0. The values of σ_n are so large that the gaussian distribution of x_n causes most points selected to lie outside the hypersphere defined by Eq. (50).

To get around the problems of both the first and the second methods of sampling, we used an arbitrarily chosen sampling distribution $\mathfrak{P}_n(x_n)$ for each of the N_c dimensions of the integral. To compensate for this method of choosing the samples, it is merely necessary to introduce a factor of $\left[\prod_1^{N_c} \mathfrak{P}_n(x_n)\right]^{-1}$ into the integrand. We choose $\mathfrak{P}_n(x_n)$ to match the gaussian distribution with variance σ_n^2 for the larger values of n in the integration, for which σ_n^2 is small and there is, therefore, no tendency to pick values of x_n that are incompatible with Eq. (50). For the smaller values of n, with larger corresponding values of σ_n^2, we chose a gaussian distribution with a variance σ_0^2. Here σ_0^2 is the same for all of the dimensions, and significantly less than the corresponding σ_n^2 values. To establish the transition between what we have called the large values of n and the small values of n, we determined a transition value, N_T, which would satisfy the requirement that

$$N_T\,\sigma_0^2 + \sum_{N_T+1}^{N_c} \sigma_n^2 = 0.9 \quad , \tag{51}$$

where

$$\sigma_0^2 = \sigma_{N_T}^2 \quad . \tag{52}$$

The value of N_T is directly obtainable from the eigenvalues of Table III, using Eq. (44).

Using a gaussian sampling distribution with variance $\sigma_0^2 = \sigma_{N_T}^2$ for variables $x_1, x_2, x_3, \ldots x_{N_T}$, and variance σ_n^2 for the variables $x_{N_T+1}, x_{N_T+2}, \ldots, x_{N_c}$, we have found that Eq. (50) is satisfied between one-third and two-thirds of the time by the hyper-space random vector so chosen. Using this sampling procedure, and noting that now the probability integral of Eq. (49) has the form

$$P_{CENSOR} = \sum_{\substack{\text{random} \\ \text{samples} \\ \{x_n\}_{N_c}}} \prod_{n=1}^{N_T} \frac{(2\pi\,\sigma_n^2)^{-1} \exp\left(-\frac{1}{2}\,x_n^2/\sigma_n^2\right)}{(2\pi\,\sigma_0^2)^{-1} \exp\left(-\frac{1}{2}\,x_n^2/\sigma_0^2\right)} \; \frac{Q(\{x_n\})}{\begin{pmatrix}\text{number of} \\ \text{samples}\end{pmatrix}} , \qquad (53)$$

where

$$Q(\{x_n\}) = \begin{cases} 0 \quad , & \text{if } \sum_{1}^{N_c} x_n^2 > 0.9 \\ \\ 1 \quad , & \text{if } \sum_{1}^{N_c} x_n^2 < 0.9 \end{cases} \quad . \qquad (54)$$

The evaluation of P_{CENSOR} was carried out in accordance with Eq. (53) using samples of 100 points for various values of D/r_0. By repeating the evaluation a number of times, it was possible to obtain an estimate not only of P_{CENSOR}, but also of the rms uncertainty in our answer. In Table IV, we list our results. These results are the basic objective of our numerical exercise. In the next section, we discuss the interpretation of these results.

Discussion of Results

The basic results are those presented in Table IV and we shall center our discussion about these results. The very large variation of P_{CENSOR} with D/r_0 is apparent from even a cursory examination. In Fig. 1, we have plotted P_{CENSOR} as a function of $(D/r_0)^2$. It is interesting to note how well the data is fit by an exponential dependence on aperture area. This is in

good agreement with an earlier conjecture by Hufnagel,[8] though the coefficients of the fit are significantly different from those suggested by Hufnagel. We find that the data is well represented by the relationship

$$P_{CENSOR} = 5.6 \exp\left[-0.1557 (D/r_0)^2\right] \quad , \tag{55}$$

at least for values of $D/r_0 \geq 5$.

It is interesting to remark that if a CENSORING experiment were performed with $D/r_0 = 15$, as would be the nominal condition for a 1.5 m telescope (with r_0 nominally equal to 0.1 m), then the probability of getting a good picture in a single short exposure would be about 3.4×10^{-15} . If independently distorted wavefront short exposures could be obtained at the rate of 100 per second, it would take more than 800 million hours "on an average" to get a good picture, i.e., one for which the average wavefront distortion over the aperture was less than one-radian. If the aperture diameter were reduced to 1 m , so that $D/r_0 = 10$, the probability would be about 1.1×10^{-6} , and the expected waiting time to get a good picture would become about 2.5 hours (if we can get 100 independent wavefront distortion samples per second). With a 0.7 m diameter aperture, the waiting time shrinks to only 3.5 seconds. Clearly, in a CENSORING experiment, it is critical to know what r_0 is and to not make the aperture diameter much larger than about $7\,r_0$, unless very long waiting times are acceptable.

We note that in certain cases, astronomical seeing with r_0 values in excess of 0.15 m have been reported.[7] In such cases, it would be quite appropriate to attempt a CENSORING experiment with a 1 m diameter aperture, but it is critical that the aperture be properly stopped, and this requires current knowledge of r_0 , and appropriate planning in the implementation of the experiment.

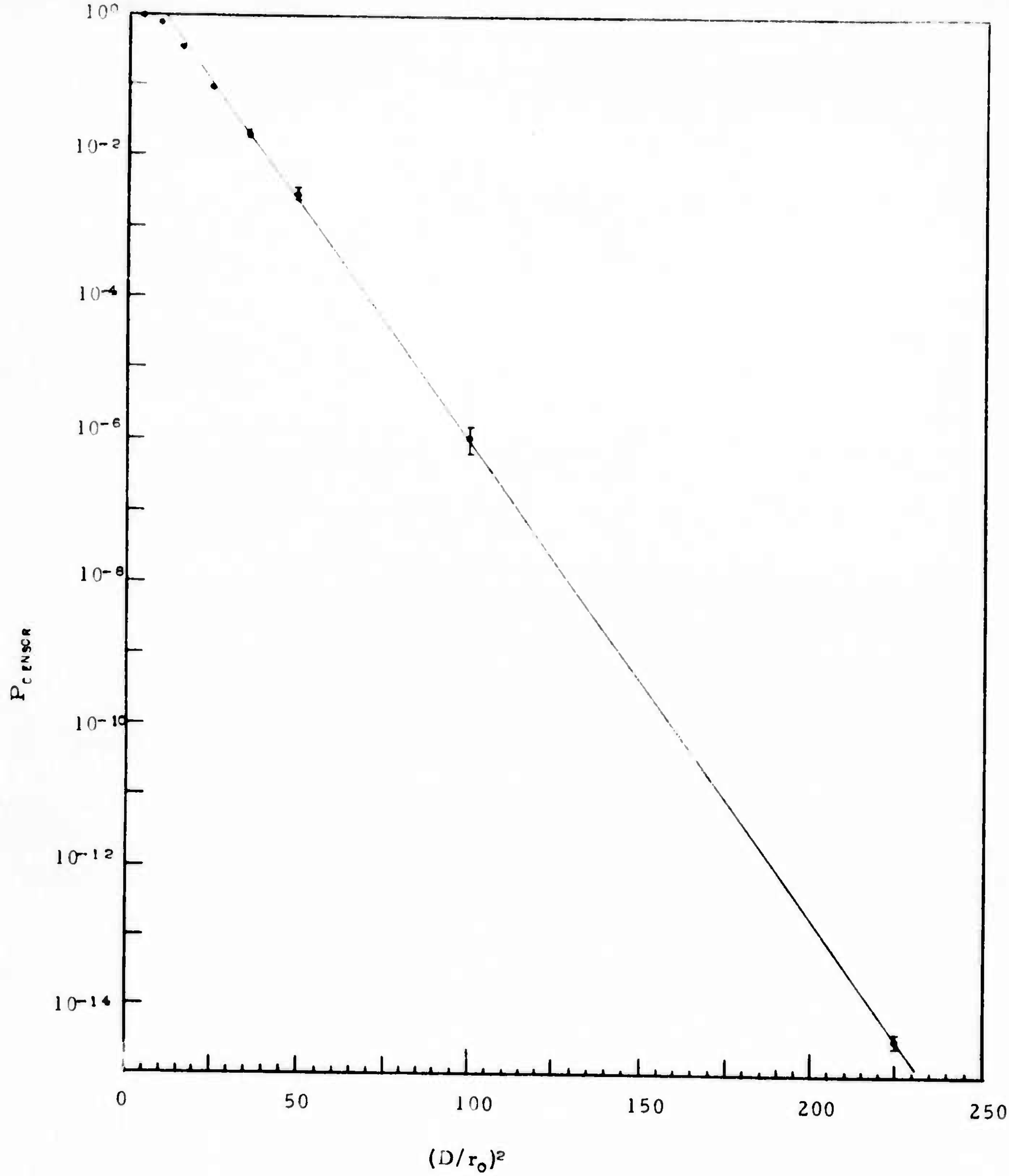

Figure 1. Probability of Obtaining a Good Short Exposure Image as a Function of Aperture Diameter. A good image is defined as one with less than one radian^2 effective wavefront error (i.e., wavefront error excluding tilt) over the aperture. Aperture diameter D is measured in units of the wavefront distortion length, r_o. P_{CENSOR} is the probability.

Table III

Eigenvalue List

Each eigenvalue and its associated values of p and q are listed, for all values covered in Table II. When $q \neq 0$, the eigenvalue is considered to be listed twice, as indicated by the nature of the overall rank-order column, N, and by the Q-value column.

N	EIGEN VALUE	CUM. SUM.	P	Q
1. 2	.018765000	.037530000	1	2. -2
3	.018733000	.056263000	1	0
4. 5	.005215500	.066694000	1	3. -3
6. 7	.005180000	.077054000	1	1. -1
8. 9	.002153400	.081360800	1	4. -4
10. 11	.001645600	.084652000	2	2. -2
12	.001632200	.086284200	2	0
13. 14	.001084800	.088453800	1	5. -5
15. 16	.000773650	.090001100	2	3. -3
17. 18	.000757140	.091515380	2	1. -1
19. 20	.000617730	.092750840	1	6. -6
21. 22	.000428150	.093607140	2	4. -4
23. 24	.000382540	.094372220	1	7. -7
25. 26	.000382130	.095136480	3	2. -2
27	.000373790	.095510270	3	0
28. 29	.000262110	.096034490	2	5. -5
30. 31	.000251910	.096538310	1	8. -8
32. 33	.000224370	.096987050	3	3. -3
34. 35	.000215100	.097417250	3	1. -1
36. 37	.000173460	.097764970	1	9. -9
38. 39	.000172040	.098109050	2	6. -6
40. 41	.000143750	.098396550	3	4. -4
42. 43	.000133880	.098664310	4	2. -2
44	.000129880	.098794190	4	0
45. 46	.000124560	.099043310	1	10.-10
47. 48	.000118870	.099281050	2	7. -7
49. 50	.000097881	.099476812	3	5. -5
51. 52	.000091963	.099660738	1	11.-11
53. 54	.000089079	.099838896	4	3. -3
55. 56	.000085452	.100009800	2	8. -8
57. 58	.000084808	.100179416	4	1. -1
59. 60	.000069736	.100318848	3	6. -6
61. 62	.000069646	.100458180	1	12.-12
63. 64	.000063398	.100584976	2	9. -9
65. 66	.000062420	.100709816	4	4. -4
67. 68	.000058784	.100827384	5	2. -2
69	.000057899	.100885283	5	0
70. 71	.000053855	.100992993	1	13.-13
72. 73	.000051459	.101095911	3	7. -7
74. 75	.000048271	.101192453	2	10.-10
76. 77	.000045524	.101283501	4	5. -5
78. 79	.000042494	.101368489	5	3. -3
80. 81	.000042428	.101453345	1	14.-14
82. 83	.000041356	.101536057	5	1. -1
84. 85	.000039058	.101614173	3	8. -8
86. 87	.000037553	.101689279	2	11.-11
88. 89	.000034265	.101757809	4	6. -6
90. 91	.000033936	.101825681	1	15.-15
92. 93	.000031680	.101889041	5	4. -4
94	.000030739	.101919780	6	0

N	EIGEN VALUE	CUM. SUM.	P	Q
95. 96	.000030339	.101980458	3	9, -9
97. 98	.000029757	.102039972	2	12,-12
99.100	.000029734	.102099440	6	2, -2
101.102	.000027540	.102154520	1	16,-16
103.104	.000026458	.102207436	4	7, -7
105.106	.000024261	.102255958	5	5, -5
107.108	.000024029	.102304016	3	10,-10
109.110	.000023952	.102351920	2	13,-13
111.112	.000023465	.102398850	6	1, -1
113.114	.000022862	.102444574	6	3, -3
115.116	.000022603	.102489780	1	17,-17
117.118	.000020867	.102531514	4	8, -8
119.120	.000019549	.102570612	2	14,-14
121.122	.000019348	.102609308	3	11,-11
123.124	.000019007	.102647322	5	6, -6
125.126	.000018773	.102684868	1	18,-18
127	.000018529	.102703397	7	0
128.129	.000017861	.102739119	6	4, -4
130.131	.000016753	.102772625	4	9, -9
132.133	.000016643	.102805911	7	2, -2
134.135	.000016146	.102838203	2	15,-15
136.137	.000015804	.102869811	3	12,-12
138.139	.000015725	.102901261	1	19,-19
140.141	.000015182	.102931625	5	7, -7
142.143	.000014838	.102961301	7	1, -1
144.145	.000014202	.102989705	6	5, -5
146.147	.000013658	.103017021	4	10,-10
148.149	.000013482	.103043985	2	16,-16
150.151	.000013446	.103070877	7	3, -3
152.153	.000013307	.103097491	1	20,-20
154.155	.000013071	.103123633	3	13,-13
156.157	.000012327	.103148287	5	8, -8
158	.000012232	.103160519	8	0
159.160	.000011478	.103183475	6	6, -6
161.162	.000011363	.103206201	2	17,-17
163.164	.000011331	.103228863	1	21,-21
165.166	.000011281	.103251425	4	11,-11
167.168	.000010931	.103273287	3	14,-14
169.170	.000010892	.103295071	7	4, -4
171.172	.000010318	.103315707	8	2, -2
173.174	.000010151	.103336009	5	9, -9
175.176	.000010135	.103356279	8	1, -1
177.178	.000009738	.103375754	1	22,-22
179.180	.000009662	.103395079	2	18,-18
181.182	.000009428	.103413934	4	12,-12
183.184	.000009415	.103432763	6	7, -7
185.186	.000009230	.103451223	3	15,-15
187.188	.000008917	.103469056	7	5, -5
189	.000008632	.103477688	9	0
190.191	.000008488	.103494665	8	3, -3

N	EIGEN VALUE	CUM. SUM.	P	Q
192,193	.000008664	.103511593	5	10,-10
194,195	.000008605	.103528403	1	23,-23
196,197	.000008278	.103544959	2	19,-19
198,199	.000007959	.103560876	4	13,-13
200,201	.000007864	.103576604	3	16,-16
202,203	.000007823	.103592250	6	8, -8
204,205	.000007456	.103607163	9	1, -1
206,207	.000007387	.103621938	7	6, -6
208,209	.000007318	.103636574	1	24,-24
210	.000007152	.103643726	10	0
211,212	.000007145	.103658016	2	20,-20
213,214	.000007134	.103672284	5	11,-11
215,216	.000007055	.103686393	8	4, -4
217,218	.000006905	.103700203	9	2, -2
219,220	.000006782	.103713766	4	14,-14
221,222	.000006751	.103727268	3	17,-17
223,224	.000006576	.103740420	6	9, -9
225,226	.000006435	.103753290	10	1, -1
227,228	.000006389	.103766067	1	25,-25
229,230	.000006205	.103778477	2	21,-21
231,232	.000006192	.103790861	7	7, -7
233	.000006166	.103797027	11	0
234,235	.000006071	.103809169	5	12,-12
236,237	.000005911	.103820990	8	5, -5
238,239	.000005840	.103832671	3	18,-18
240,241	.000005825	.103844321	4	15,-15
242,243	.000005671	.103855662	9	3, -3
244,245	.000005625	.103866913	1	26,-26
246,247	.000005585	.103878082	6	10,-10
248,249	.000005520	.103889123	11	1, -1
250,251	.000005423	.103899970	2	22,-22
252,253	.000005245	.103910459	7	8, -8
254,255	.000005211	.103920881	5	13,-13
256,257	.000005083	.103931048	3	19,-19
258,259	.000005042	.103941132	4	16,-16
260,261	.000005000	.103951132	8	6, -6
262	.000004975	.103956106	12	0
263,264	.000004959	.103966023	1	27,-27
265,266	.000004895	.103975813	10	2, -2
267,268	.000004800	.103985413	9	4, -4
269,270	.000004786	.103994985	6	11,-11
271,272	.000004764	.104004512	2	23,-23
273,274	.000004509	.104013530	5	14,-14
275,276	.000004485	.104022499	7	9, -9
277,278	.000004454	.104031406	3	20,-20
279,280	.000004409	.104040224	1	28,-28
281,282	.000004392	.104049009	4	17,-17
283,284	.000004267	.104057543	8	7, -7
285,286	.000004236	.104066015	12	1, -1
287,288	.000004208	.104074432	2	24,-24

N	EIGEN VALUE	CUM. SUM.	P	J
289,290	.000004136	.104082703	6	12,-12
291,292	.000004097	.104090897	9	5,-5
293,294	.000003960	.104098818	10	3,-3
295,296	.000003926	.104106673	5	15,-15
297,298	.000003921	.104114515	3	21,-21
299,300	.000003919	.104122352	1	29,-29
301,302	.000003869	.104130090	7	10,-10
303,304	.000003852	.104137793	4	18,-18
305	.000003811	.104141604	13	0
306,307	.000003733	.104149070	2	25,-25
308,309	.000003674	.104156418	8	8,-8
310,311	.000003627	.104163671	13	1,-1
312,313	.000003600	.104170871	6	13,-13
314,315	.000003526	.104177924	9	6,-6
316,317	.000003514	.104184953	1	30,-30
318,319	.000003480	.104191914	11	2,-2
320,321	.000003473	.104198859	3	22,-22
322,323	.000003444	.104205748	5	16,-16
324,325	.000003396	.104212540	10	4,-4
326,327	.000003396	.104219331	4	19,-19
328,329	.000003363	.104226057	7	11,-11
330,331	.000003328	.104232713	2	26,-26
332	.000003294	.104236007	14	0
333,334	.000003189	.104242385	8	9,-9
335,336	.000003156	.104248696	6	14,-14
337,338	.000003146	.104254989	1	31,-31
339,340	.000003087	.104261163	3	23,-23
341,342	.000003058	.104267278	9	7,-7
343,344	.000003038	.104273354	5	17,-17
345,346	.000003011	.104279375	4	20,-20
347,348	.000002977	.104285329	2	27,-27
349,350	.000002948	.104291225	10	5,-5
351,352	.000002945	.104297114	7	12,-12
353,354	.000002870	.104302854	11	3,-3
355,356	.000002844	.104308541	1	32,-32
357,358	.000002789	.104314119	8	10,-10
359,360	.000002782	.104319684	6	15,-15
361,362	.000002760	.104325204	3	24,-24
363,364	.000002695	.104330593	5	18,-18
365,366	.000002686	.104335966	12	2,-2
367,368	.000002682	.104341329	14	1,-1
369,370	.000002681	.104346691	4	21,-21
371,372	.000002676	.104352043	2	28,-28
373,374	.000002671	.104357385	9	8,-8
375,376	.000002595	.104362575	7	13,-13
377,378	.000002576	.104367728	10	6,-6
379,380	.000002561	.104372851	1	33,-33
381	.000002499	.104375350	15	0
382,383	.000002492	.104380333	11	4,-4
384,385	.000002474	.104385282	3	25,-25

N	EIGEN VALUE	CUM. SUM.	P	Q
386,387	.000002468	.104390218	6	16,-16
388,389	.000002456	.104395130	8	11,-11
390,391	.000002411	.104399952	2	29,-29
392,393	.000002402	.104404756	5	19,-19
394,395	.000002400	.104409555	4	22,-22
396,397	.000002350	.104414255	9	9, -9
398,399	.000002331	.104418918	1	34,-34
400,401	.000002301	.104423520	7	14,-14
402,403	.000002277	.104428073	15	1, -1
404,405	.000002267	.104432606	10	7, -7
406,407	.000002231	.104437067	3	26,-26
408,409	.000002210	.104441487	13	2, -2
410,411	.000002200	.104445888	6	17,-17
412,413	.000002194	.104450276	11	5, -5
414,415	.000002183	.104454643	2	30,-30
416,417	.000002176	.104458994	8	12,-12
418,419	.000002170	.104463334	12	3, -3
420,421	.000002156	.104467645	4	23,-23
422,423	.000002151	.104471948	5	20,-20
424,425	.000002112	.104476171	1	35,-35
426,427	.000002081	.104480332	9	10,-10
428	.000002071	.104482403	16	0
429,430	.000002051	.104486506	7	15,-15
431,432	.000002015	.104490535	3	27,-27
433,434	.000002007	.104494548	10	8, -8
435,436	.000001980	.104498508	2	31,-31
437,438	.000001972	.104502452	6	18,-18
439,440	.000001945	.104506343	4	24,-24
441,442	.000001944	.104510231	11	6, -6
443,444	.000001939	.104514109	8	13,-13
445,446	.000001935	.104517979	5	21,-21
447,448	.000001935	.104521848	1	36,-36
449,450	.000001894	.104525637	12	4, -4
451,452	.000001854	.104529344	9	11,-11
453,454	.000001838	.104533021	7	16,-16
455,456	.000001829	.104536680	3	28,-28
457,458	.000001816	.104540312	16	1, -1
459,460	.000001804	.104543921	2	32,-32
461,462	.000001787	.104547495	10	9, -9
463,464	.000001775	.104551045	6	19,-19
465,466	.000001761	.104554567	4	25,-25
467,468	.000001760	.104558088	1	37,-37
469,470	.000001748	.104561584	5	22,-22
471,472	.000001743	.104565070	14	2, -2
473,474	.000001738	.104568545	8	14,-14
475,476	.000001733	.104572011	11	7, -7
477	.000001726	.104573737	17	0
478,479	.000001709	.104577156	13	3, -3
480,481	.000001687	.104580529	12	5, -5
482,483	.000001663	.104583855	3	29,-29

V	EIGEN VALUE	CUM. SUM.	P	Q
484,485	.000001661	.104587177	9	12,-12
486,487	.000001655	.104590487	7	17,-17
488,489	.000001646	.104593780	2	33,-33
490,491	.000001622	.104597024	1	38,-38
492,493	.000001604	.104600233	6	20,-20
494,495	.000001602	.104603436	10	10,-10
496,497	.000001601	.104606638	4	26,-26
498,499	.000001584	.104609807	5	23,-23
500,501	.000001565	.104612937	8	15,-15
502,503	.000001554	.104616045	11	8, -8
504,505	.000001520	.104619084	3	30,-30
506,507	.000001512	.104622109	12	6, -6
508,509	.000001509	.104625126	2	34,-34
510,511	.000001497	.104628121	7	18,-18
512,513	.000001496	.104631113	9	13,-13
514,515	.000001487	.104634087	13	4, -4
516,517	.000001482	.104637052	1	39,-39
518,519	.000001459	.104639970	4	27,-27
520,521	.000001456	.104642882	6	21,-21
522,523	.000001443	.104645768	10	11,-11
524,525	.000001442	.104648653	5	24,-24
526,527	.000001416	.104651486	8	16,-16
528,529	.000001404	.104654293	15	2, -2
530,531	.000001401	.104657095	11	9, -9
532,533	.000001389	.104659874	3	31,-31
534,535	.000001389	.104662652	17	1, -1
536,537	.000001384	.104665420	2	35,-35
538,539	.000001375	.104668170	14	3, -3
540,541	.000001374	.104670918	1	40,-40
542,543	.000001371	.104673660	16	2, -2
544,545	.000001365	.104676390	12	7, -7
546,547	.000001360	.104679110	7	19,-19
548,549	.000001355	.104681819	9	14,-14
550,551	.000001335	.104684489	4	28,-28
552,553	.000001334	.104687158	13	5, -5
554,555	.000001327	.104689811	6	22,-22
556,557	.000001317	.104692444	5	25,-25
558,559	.000001307	.104695058	10	12,-12
560,561	.000001287	.104697633	8	17,-17
562,563	.000001278	.104700188	3	32,-32
564,565	.000001275	.104702739	2	36,-36
566,567	.000001270	.104705278	11	10,-10
568,569	.000001260	.104707798	1	41,-41
570	.000001259	.104709057	18	0
571,572	.000001240	.104711537	7	20,-20
573,574	.000001239	.104714016	12	8, -8
575,576	.000001232	.104716480	9	15,-15
577,578	.000001224	.104718928	4	29,-29
579,580	.000001213	.104721354	6	23,-23
581,582	.000001212	.104723778	13	6, -6

V	EIGEN VALUE	CUM. SUM.	P	Q
583.584	.000001207	.104726192	5	26, -26
585.586	.000001203	.104728597	14	4, -4
587.588	.000001190	.104730977	10	13, -13
589.590	.000001175	.104733327	8	18, -18
591.592	.000001175	.104735676	2	37, -37
593.594	.000001174	.104738024	3	33, -33
595.596	.000001157	.104740338	11	11, -11
597.598	.000001135	.104742609	7	21, -21
599.600	.000001133	.104744875	15	3, -3
601.602	.000001131	.104747137	12	9, -9
603.604	.000001127	.104749390	4	30, -30
605.606	.000001126	.104751642	9	16, -16
607.608	.000001113	.104753867	6	24, -24
609.610	.000001108	.104756084	5	27, -27
611.612	.000001108	.104758300	13	7, -7
613.614	.000001090	.104760479	14	5, -5
615.616	.000001088	.104762656	10	14, -14
617.618	.000001087	.104764831	2	38, -38
619.620	.000001085	.104767001	3	34, -34
621.622	.000001077	.104769155	8	19, -19
623.624	.000001060	.104771275	11	12, -12
625.626	.000001043	.104773361	7	22, -22
627.628	.000001039	.104775439	4	31, -31
629.630	.000001038	.104777515	12	10, -10
631.632	.000001033	.104779580	9	17, -17
633.634	.000001024	.104781628	6	25, -25
635.636	.000001024	.104783676	17	2, -2
637.638	.000001022	.104785720	5	28, -28
639.640	.000001019	.104787758	13	8, -8
641.642	.000001007	.104789773	16	3, -3
643.644	.000001006	.104791786	2	39, -39
645.646	.000001002	.104793790	3	35, -35
647.648	.000001002	.104795793	14	6, -6
649.650	.000001000	.104797794	15	4, -4
651.652	.000001000	.104799793	10	15, -15
653.654	.000000992	.104801776	18	1, -1
655.656	.000000991	.104803758	8	20, -20
657.658	.000000975	.104805709	11	13, -13
659.660	.000000961	.104807631	4	32, -32
661.662	.000000961	.104809554	7	23, -23
663.664	.000000957	.104811468	12	11, -11
665.666	.000000952	.104813372	9	18, -18
667.668	.000000946	.104815263	6	26, -26
669.670	.000000944	.104817150	5	29, -29
671.672	.000000943	.104819036	13	9, -9
673.674	.000000936	.104820907	2	40, -40
675	.000000935	.104821842	19	0
676.677	.000000931	.104823704	3	36, -36
678.679	.000000929	.104825563	14	7, -7
680.681	.000000923	.104827408	10	16, -16

N	EIGEN VALUE	CUM. SUM.	P	Q
682.683	.000000918	.104829244	15	5, -5
684.685	.000000915	.104831073	8	21,-21
686.687	.000000902	.104832877	11	14,-14
688.689	.000000891	.104834658	4	33,-33
690.691	.000000889	.104836436	7	24,-24
692.693	.000000887	.104838211	12	12,-12
694.695	.000000880	.104839971	9	19,-19
696.697	.000000877	.104841726	19	1, -1
698.699	.000000876	.104843478	13	10,-10
700.701	.000000876	.104845231	17	3, -3
702.703	.000000876	.104846982	16	4, -4
704.705	.000000875	.104848733	5	30,-30
706.707	.000000875	.104850483	6	27,-27
708.709	.000000869	.104852222	2	41,-41
710.711	.000000867	.104853955	14	8, -8
712.713	.000000863	.104855681	3	37,-37
714.715	.000000863	.104857407	18	2, -2
716.717	.000000857	.104859121	15	6, -6
718.719	.000000855	.104860832	10	17,-17
720.721	.000000848	.104862527	8	22,-22
722.723	.000000838	.104864203	11	15,-15
724.725	.000000828	.104865860	4	34,-34
726.727	.000000827	.104867513	12	13,-13
728.729	.000000825	.104869162	7	25,-25
730.731	.000000819	.104870801	13	11,-11
732.733	.000000817	.104872436	9	20,-20
734.735	.000000813	.104874062	14	9, -9
736.737	.000000813	.104875688	6	28,-28
738.739	.000000812	.104877313	5	31,-31
740.741	.000000811	.104878934	16	5, -5
742.743	.000000808	.104880549	15	7, -7
744.745	.000000806	.104882160	3	38,-38
746.747	.000000796	.104883752	10	18,-18
748.749	.000000788	.104885327	8	23,-23
750.751	.000000785	.104886897	17	4, -4
752.753	.000000782	.104888461	11	16,-16
754.755	.000000774	.104890010	12	14,-14
756.757	.000000771	.104891551	4	35,-35
758.759	.000000771	.104893093	13	12,-12
760.761	.000000769	.104894631	14	10,-10
762.763	.000000768	.104896166	7	26,-26
764.765	.000000767	.104897699	15	8, -8
766.767	.000000766	.104899232	18	3, -3
768.769	.000000765	.104900761	19	2, -2
770.771	.000000763	.104902288	16	6, -6
772.773	.000000761	.104903810	9	21,-21
774.775	.000000757	.104905325	5	32,-32
776.777	.000000757	.104906838	6	29,-29
778.779	.000000750	.104908338	3	39,-39
780.781	.000000744	.104909825	10	19,-19

N	EIGEN VALUE	CUM. SUM.	P	Q
782,783	.000000739	.104911302	17	5, -5
784,785	.000000737	.104912777	16	7, -7
786,787	.000000735	.104914247	15	9, -9
788,789	.000000735	.104915717	8	24,-24
790,791	.000000733	.104917184	11	17,-17
792,793	.000000731	.104918646	14	11,-11
794,795	.000000728	.104920103	12	15,-15
796,797	.000000728	.104921559	13	13,-13
798,799	.000000728	.104923015	18	4, -4
800,801	.000000720	.104924455	4	36,-36
802,803	.000000716	.104925886	7	27,-27
804,805	.000000715	.104927316	17	6, -6
806,807	.000000712	.104928740	9	22,-22
808,809	.000000706	.104930153	6	30,-30
810,811	.000000706	.104931565	5	33,-33
812,813	.000000705	.104932975	16	8, -8
814,815	.000000703	.104934381	3	40,-40
816,817	.000000697	.104935775	10	20,-20
818,819	.000000697	.104937169	15	10,-10
820,821	.000000691	.104938552	14	12,-12
822,823	.000000689	.104939931	11	18,-18
824,825	.000000688	.104941306	13	14,-14
826,827	.000000687	.104942680	8	25,-25
828,829	.000000687	.104944053	12	16,-16
830,831	.000000673	.104945399	4	37,-37
832,833	.000000670	.104946739	7	28,-28
834,835	.000000668	.104948074	9	23,-23
836,837	.000000661	.104949397	5	34,-34
838,839	.000000661	.104950718	6	31,-31
840,841	.000000656	.104952030	3	41,-41
842,843	.000000656	.104953342	10	21,-21
844,845	.000000649	.104954639	11	19,-19
846,847	.000000646	.104955931	12	17,-17
848,849	.000000645	.104957220	8	26,-26
850,851	.000000645	.104958509	13	15,-15
852,853	.000000644	.104959797	14	13,-13
854,855	.000000644	.104961084	15	11,-11
856,857	.000000643	.104962371	16	9, -9
858,859	.000000642	.104963655	17	7, -7
860,861	.000000640	.104964935	18	5, -5
862,863	.000000631	.104966197	4	38,-38
864,865	.000000631	.104967459	19	3, -3
866,867	.000000628	.104968715	7	29,-29
868,869	.000000628	.104969970	9	24,-24
870,871	.000000620	.104971210	6	32,-32
872,873	.000000619	.104972448	5	35,-35
874,875	.000000617	.104973682	10	22,-22
876,877	.000000610	.104974903	11	20,-20
878,879	.000000606	.104976114	8	27,-27
880,881	.000000605	.104977323	12	18,-18

N	EIGEN VALUE	CUM. SUM.	P	Q
882,883	.000000599	.104978521	13	16,-16
884,885	.000000592	.104979705	4	39,-39
886,887	.000000591	.104980887	14	14,-14
888,889	.000000591	.104982069	9	25,-25
890,891	.000000591	.104983251	7	30,-30
892	.000000588	.104983839	20	0
893,894	.000000583	.104985004	5	36,-36
895,896	.000000582	.104986169	6	33,-33
897,898	.000000581	.104987331	15	12,-12
899,900	.000000581	.104988492	10	23,-23
901,902	.000000572	.104989636	11	21,-21
903,904	.000000571	.104990778	8	28,-28
905,906	.000000566	.104991911	16	10,-10
907,908	.000000563	.104993037	12	19,-19
909,910	.000000557	.104994151	4	40,-40
911,912	.000000557	.104995265	9	26,-26
913,914	.000000556	.104996378	7	31,-31
915,916	.000000552	.104997481	13	17,-17
917,918	.000000549	.104998578	6	34,-34
919,920	.000000548	.104999673	5	37,-37
921,922	.000000546	.105000765	10	24,-24
923,924	.000000543	.105001851	17	8,-8
925,926	.000000541	.105002932	20	1,-1
927,928	.000000538	.105004008	8	29,-29
929,930	.000000537	.105005082	14	15,-15
931,932	.000000535	.105006152	11	22,-22
933,934	.000000525	.105007203	7	32,-32
935,936	.000000525	.105008252	9	27,-27
937,938	.000000525	.105009301	4	41,-41
939,940	.000000522	.105010344	12	20,-20
941,942	.000000517	.105011379	15	13,-13
943,944	.000000517	.105012414	6	35,-35
945,946	.000000517	.105013448	5	38,-38
947,948	.000000512	.105014472	10	25,-25
949,950	.000000506	.105015489	8	30,-30
951,952	.000000505	.105016500	13	18,-18
953,954	.000000501	.105017502	18	6,-6
955,956	.000000498	.105018498	11	23,-23
957,958	.000000496	.105019490	7	33,-33
959,960	.000000494	.105020479	9	28,-28
961,962	.000000489	.105021457	16	11,-11
963,964	.000000489	.105022436	6	36,-36
965,966	.000000488	.105023411	5	39,-39
967,968	.000000484	.105024380	14	16,-16
969,970	.000000482	.105025343	12	21,-21
971,972	.000000480	.105026303	10	26,-26
973,974	.000000480	.105027262	8	31,-31
975,976	.000000469	.105028201	7	34,-34
977,978	.000000465	.105029132	9	29,-29
979,980	.000000463	.105030058	11	24,-24

N	EIGEN VALUE	CUM. SUM.	P	Q
981.982	.000000463	.105030984	6	37,-37
983.984	.000000462	.105031908	5	40,-40
985.986	.000000461	.105032830	13	19,-19
987.988	.000000457	.105033743	15	14,-14
989.990	.000000453	.105034650	8	32,-32
991.992	.000000449	.105035547	10	27,-27
993.994	.000000447	.105036442	17	9, -9
995.996	.000000444	.105037330	7	35,-35
997.998	.000000444	.105038217	12	22,-22
999.1000	.000000439	.105039094	6	38,-38
1001.1002	.000000437	.105039969	9	30,-30
1003.1004	.000000437	.105040843	5	41,-41
1005.1006	.000000435	.105041713	14	17,-17
1007.1008	.000000430	.105042573	11	25,-25
1009.1010	.000000428	.105043429	8	33,-33
1011.1012	.000000421	.105044270	7	36,-36
1013.1014	.000000420	.105045110	10	28,-28
1015.1016	.000000419	.105045949	13	20,-20
1017.1018	.000000419	.105046786	16	12,-12
1019.1020	.000000416	.105047617	6	39,-39
1021.1022	.000000411	.105048439	9	31,-31
1023.1024	.000000410	.105049260	19	4, -4
1025.1026	.000000408	.105050076	12	23,-23
1027.1028	.000000404	.105050884	8	34,-34
1029.1030	.000000401	.105051686	15	15,-15
1031.1032	.000000399	.105052485	11	26,-26
1033.1034	.000000398	.105053281	7	37,-37
1035.1036	.000000395	.105054071	6	40,-40
1037.1038	.000000392	.105054855	10	29,-29
1039.1040	.000000390	.105055634	14	18,-18
1041.1042	.000000386	.105056406	9	32,-32
1043.1044	.000000381	.105057169	8	35,-35
1045.1046	.000000381	.105057931	13	21,-21
1047.1048	.000000378	.105058688	18	7, -7
1049.1050	.000000378	.105059443	7	38,-38
1051.1052	.000000375	.105060193	6	41,-41
1053.1054	.000000375	.105060943	12	24,-24
1055.1056	.000000370	.105061682	11	27,-27
1057.1058	.000000366	.105062414	10	30,-30
1059.1060	.000000365	.105063144	17	10,-10
1061.1062	.000000363	.105063869	9	33,-33
1063.1064	.000000360	.105064589	8	36,-36
1065.1066	.000000358	.105065304	7	39,-39
1067.1068	.000000357	.105066018	16	13,-13
1069.1070	.000000352	.105066722	15	16,-16
1071.1072	.000000349	.105067420	14	19,-19
1073.1074	.000000346	.105068112	13	22,-22
1075.1076	.000000344	.105068801	12	25,-25
1077.1078	.000000343	.105069486	11	28,-28
1079.1080	.000000342	.105070169	10	31,-31

V	EIGEN VALUE	CUM. SUM.	P	Q
1081.1082	.000000341	.105070851	9	34,-34
1083.1084	.000000340	.105071530	8	37,-37
1085.1086	.000000339	.105072209	7	40,-40
1087.1088	.000000321	.105072852	7	41,-41
1089.1090	.000000321	.105073494	8	38,-38
1091.1092	.000000320	.105074133	9	35,-35
1093.1094	.000000319	.105074772	10	32,-32
1095.1096	.000000318	.105075407	11	29,-29
1097.1098	.000000316	.105076040	12	26,-26
1099.1100	.000000315	.105076669	13	23,-23
1101.1102	.000000312	.105077294	14	20,-20
1103.1104	.000000309	.105077912	15	17,-17
1105.1106	.000000304	.105078521	16	14,-14
1107.1108	.000000303	.105079126	8	39,-39
1109.1110	.000000301	.105079728	9	36,-36
1111.1112	.000000298	.105080324	10	33,-33
1113.1114	.000000297	.105080918	17	11,-11
1115.1116	.000000295	.105081508	11	30,-30
1117.1118	.000000291	.105082090	12	27,-27
1119.1120	.000000286	.105082663	8	40,-40
1121.1122	.000000286	.105083235	13	24,-24
1123.1124	.000000285	.105083805	18	8, -8
1125.1126	.000000283	.105084370	9	37,-37
1127.1128	.000000280	.105084930	14	21,-21
1129.1130	.000000279	.105085487	10	34,-34
1131.1132	.000000274	.105086035	11	31,-31
1133.1134	.000000272	.105086578	15	18,-18
1135.1136	.000000270	.105087119	8	41,-41
1137.1138	.000000266	.105087654	12	28,-28
1139.1140	.000000266	.105088187	9	38,-38
1141.1142	.000000260	.105088708	13	25,-25
1143.1144	.000000260	.105089229	10	35,-35
1145.1146	.000000260	.105089749	16	15,-15
1147.1148	.000000258	.105090265	19	5, -5
1149.1150	.000000254	.105090774	11	32,-32
1151.1152	.000000251	.105091276	14	22,-22
1153.1154	.000000250	.105091777	9	39,-39
1155.1156	.000000247	.105092270	12	29,-29
1157.1158	.000000244	.105092758	10	36,-36
1159.1160	.000000244	.105093245	17	12,-12
1161.1162	.000000239	.105093724	15	19,-19
1163.1164	.000000238	.105094200	13	26,-26
1165.1166	.000000236	.105094672	11	33,-33
1167.1168	.000000235	.105095144	9	40,-40
1169.1170	.000000228	.105095601	10	37,-37
1171.1172	.000000228	.105096057	12	30,-30
1173.1174	.000000226	.105096509	14	23,-23
1175.1176	.000000224	.105096956	16	16,-16
1177.1178	.000000222	.105097400	9	41,-41
1179.1180	.000000220	.105097841	11	34,-34

N	EIGEN VALUE	CUM. SUM.	P	Q
1181.1182	.000000217	.105098274	13	27,-27
1183.1184	.000000217	.105098708	18	9, -9
1185.1186	.000000214	.105099136	10	38,-38
1187.1188	.000000212	.105099560	15	20,-20
1189.1190	.000000210	.105099980	12	31,-31
1191.1192	.000000205	.105100391	11	35,-35
1193.1194	.000000204	.105100799	14	24,-24
1195.1196	.000000201	.105101201	10	39,-39
1197.1198	.000000201	.105101603	17	13,-13
1199.1200	.000000199	.105102000	13	28,-26
1201.1202	.000000195	.105102389	12	32,-32
1203.1204	.000000193	.105102775	16	17,-17
1205.1206	.000000192	.105103158	11	36,-36
1207.1208	.000000189	.105103536	10	40,-40
1209.1210	.000000188	.105103912	15	21,-21
1211.1212	.000000184	.105104281	14	25,-25
1213.1214	.000000182	.105104645	13	29,-29
1215.1216	.000000180	.105105005	12	33,-33
1217.1218	.000000179	.105105363	11	37,-37
1219.1220	.000000178	.105105718	10	41,-41
1221.1222	.000000167	.105106053	11	38,-38
1223.1224	.000000167	.105106388	12	34,-34
1225.1226	.000000167	.105106722	13	30,-30
1227.1228	.000000167	.105107057	14	26,-26
1229.1230	.000000167	.105107392	15	22,-22
1231.1232	.000000167	.105107726	16	18,-18
1233.1234	.000000167	.105108061	17	14,-14
1235.1236	.000000167	.105108395	18	10,-10
1237.1238	.000000167	.105108729	19	6, -6
1239.1240	.000000166	.105109062	20	2, -2
1241.1242	.000000157	.105109375	11	39,-39
1243.1244	.000000155	.105109686	12	35,-35
1245.1246	.000000154	.105109993	13	31,-31
1247.1248	.000000152	.105110297	14	27,-27
1249.1250	.000000149	.105110596	15	23,-23
1251.1252	.000000147	.105110890	11	40,-40
1253.1254	.000000146	.105111182	16	19,-19
1255.1256	.000000145	.105111672	12	36,-36
1257.1258	.000000142	.105111756	13	32,-32
1259.1260	.000000140	.105112037	17	15,-15
1261.1262	.000000139	.105112314	14	28,-28
1263.1264	.000000138	.105112589	11	41,-41
1265.1266	.000000135	.105112859	12	37,-37
1267.1268	.000000134	.105113127	15	24,-24
1269.1270	.000000131	.105113390	18	11,-11
1271.1272	.000000131	.105113652	13	33,-33
1273.1274	.000000128	.105113908	16	20,-20
1275.1276	.000000127	.105114161	14	29,-29
1277.1278	.000000126	.105114613	12	38,-38
1279.1280	.000000122	.105114656	13	34,-34

N	EIGEN VALUE	CUM. SUM.	P	Q
1281.1282	.000000121	.105114897	15	25.-25
1283.1284	.000000119	.105115135	17	16.-16
1285.1286	.000000118	.105115370	12	39.-39
1287.1288	.000000116	.105115602	14	30.-30
1289.1290	.000000113	.105115828	19	7, -7
1291.1292	.000000113	.105116054	16	21.-21
1293.1294	.000000113	.105116279	13	35.-35
1295.1296	.000000110	.105116500	12	40.-40
1297.1298	.000000109	.105116718	15	26.-26
1299.1300	.000000106	.105116930	14	31.-31
1301.1302	.000000105	.105117140	18	12.-12
1303.1304	.000000105	.105117350	13	36.-36
1305.1306	.000000103	.105117556	12	41.-41
1307.1308	.000000102	.105117759	17	17.-17
1309.1310	.000000100	.105117959	16	22.-22
1311.1312	.000000099	.105118156	15	27.-27
1313.1314	.000000098	.105118352	14	32.-32
1315.1316	.000000097	.105118547	13	37.-37
1317.1318	.000000091	.105118729	13	38.-38
1319.1320	.000000090	.105118909	14	33.-33
1321.1322	.000000090	.105119089	15	28.-28
1323.1324	.000000089	.105119266	16	23.-23
1325.1326	.000000087	.105119441	17	18.-18
1327.1328	.000000085	.105119611	18	13.-13
1329.1330	.000000085	.105119781	13	39.-39
1331.1332	.000000084	.105119948	14	34.-34
1333.1334	.000000082	.105120111	15	29.-29
1335.1336	.000000080	.105120271	19	8, -8
1337.1338	.000000079	.105120430	13	40.-40
1339.1340	.000000079	.105120589	16	24.-24
1341.1342	.000000077	.105120743	14	35.-35
1343.1344	.000000076	.105120895	17	19.-19
1345.1346	.000000075	.105121045	15	30.-30
1347.1348	.000000074	.105121193	13	41.-41
1349.1350	.000000072	.105121337	14	36.-36
1351.1352	.000000071	.105121479	16	25.-25
1353.1354	.000000070	.105121619	18	14.-14
1355.1356	.000000069	.105121756	15	31.-31
1357.1358	.000000067	.105121890	14	37.-37
1359.1360	.000000066	.105122022	17	20.-20
1361.1362	.000000064	.105122151	16	26.-26
1363.1364	.000000063	.105122277	15	32.-32
1365.1366	.000000062	.105122401	14	38.-38
1367.1368	.000000061	.105122523	20	3, -3
1369.1370	.000000058	.105122640	19	9, -9
1371.1372	.000000058	.105122756	18	15.-15
1373.1374	.000000058	.105122872	17	21.-21
1375.1376	.000000058	.105122989	16	27.-27
1377.1378	.000000058	.105123105	15	33.-33
1379.1380	.000000058	.105123221	14	39.-39

N	EIGEN VALUE	CUM. SUM.	P	Q
1381.1382	.000000056	.105123329	14	40,-40
1383.1384	.000000054	.105123437	15	34,-34
1385.1386	.000000053	.105123542	16	28,-28
1387.1388	.000000051	.105123645	17	22,-22
1389.1390	.000000051	.105123747	14	41,-41
1391.1392	.000000050	.105123846	15	35,-35
1393.1394	.000000049	.105123944	18	16,-16
1395.1396	.000000048	.105124040	16	29,-29
1397.1398	.000000046	.105124132	15	36,-36
1399.1400	.000000046	.105124224	17	23,-23
1401.1402	.000000044	.105124312	19	10,-10
1403.1404	.000000044	.105124400	16	30,-30
1405.1406	.000000043	.105124485	15	37,-37
1407.1408	.000000042	.105124569	18	17,-17
1409.1410	.000000041	.105124650	17	24,-26
1411.1412	.000000040	.105124731	16	31,-31
1413.1414	.000000040	.105124811	15	38,-38
1415.1416	.000000037	.105124885	15	39,-39
1417.1418	.000000037	.105124959	16	32,-32
1419.1420	.000000037	.105125032	17	25,-25
1421.1422	.000000036	.105125104	18	18,-18
1423.1424	.000000035	.105125173	15	40,-40
1425.1426	.000000034	.105125242	19	11,-11
1427.1428	.000000034	.105125310	16	33,-33
1429.1430	.000000033	.105125376	17	26,-26
1431.1432	.000000033	.105125441	15	41,-41
1433.1434	.000000031	.105125504	16	34,-34
1435.1436	.000000031	.105125566	18	19,-19
1437.1438	.000000030	.105125625	17	27,-27
1439.1440	.000000029	.105125683	16	35,-35
1441.1442	.000000028	.105125739	20	4, -4
1443.1444	.000000027	.105125794	19	12,-12
1445.1446	.000000027	.105125848	18	20,-20
1447.1448	.000000027	.105125902	16	36,-36
1449.1450	.000000027	.105125956	17	28,-28
1451.1452	.000000025	.105126006	16	37,-37
1453.1454	.000000025	.105126055	17	29,-29
1455.1456	.000000024	.105126103	18	21,-21
1457.1458	.000000023	.105126149	16	38,-36
1459.1460	.000000022	.105126194	17	30,-30
1461.1462	.000000022	.105126238	19	13,-13
1463.1464	.000000022	.105126282	16	39,-39
1465.1466	.000000021	.105126324	18	22,-22
1467.1468	.000000021	.105126365	17	31,-31
1469.1470	.000000020	.105126406	16	40,-40
1471.1472	.000000019	.105126444	16	41,-41
1473.1474	.000000019	.105126482	17	32,-32
1475.1476	.000000019	.105126519	18	23,-23
1477.1478	.000000018	.105126555	19	14,-14
1479.1480	.000000017	.105126590	17	33,-33

N	EIGEN VALUE	CUM. SUM.	P	Q
1481,1482	.000000017	.105126623	18	24,-24
1483,1484	.000000016	.105126655	17	34,-34
1485,1486	.000000015	.105126686	20	5, -5
1487,1488	.000000015	.105126715	19	15,-15
1489,1490	.000000015	.105126745	18	25,-25
1491,1492	.000000015	.105126775	17	35,-35
1493,1494	.000000014	.105126803	17	36,-36
1495,1496	.000000013	.105126829	18	26,-26
1497,1498	.000000013	.105126855	17	37,-37
1499,1500	.000000013	.105126880	19	16,-16
1501,1502	.000000012	.105126905	18	27,-27
1503,1504	.000000012	.105126928	17	38,-38
1505,1506	.000000011	.105126951	17	39,-39
1507,1508	.000000011	.105126973	18	28,-28
1509,1510	.000000011	.105126994	19	17,-17
1511,1512	.000000010	.105127015	17	40,-40
1513,1514	.000000010	.105127035	18	29,-29
1515,1516	.000000010	.105127055	17	41,-41
1517,1518	.000000009	.105127073	20	6, -6
1519,1520	.000000009	.105127092	19	18,-18
1521,1522	.000000009	.105127110	18	30,-30
1523,1524	.000000008	.105127127	18	31,-31
1525,1526	.000000008	.105127143	19	19,-19
1527,1528	.000000008	.105127158	18	32,-32
1529,1530	.000000007	.105127172	18	33,-33
1531,1532	.000000007	.105127186	19	20,-20
1533,1534	.000000007	.105127199	18	34,-34
1535,1536	.000000006	.105127212	20	7, -7
1537,1538	.000000006	.105127224	19	21,-21
1539,1540	.000000006	.105127236	18	35,-35
1541,1542	.000000006	.105127247	18	36,-36
1543,1544	.000000005	.105127258	19	22,-22
1545,1546	.000000005	.105127268	18	37,-37
1547,1548	.000000005	.105127278	18	38,-38
1549,1550	.000000005	.105127288	19	23,-23
1551,1552	.000000005	.105127297	18	39,-39
1553,1554	.000000004	.105127305	20	8, -8
1555,1556	.000000004	.105127314	18	40,-40
1557,1558	.000000004	.105127322	19	24,-24
1559,1560	.000000004	.105127330	18	41,-41
1561,1562	.000000004	.105127338	19	25,-25
1563,1564	.000000003	.105127345	19	26,-26
1565,1566	.000000003	.105127351	20	9, -9
1567,1568	.000000003	.105127357	19	27,-27
1569,1570	.000000003	.105127363	19	28,-28
1571,1572	.000000003	.105127368	19	29,-29
1573,1574	.000000002	.105127373	20	10,-10
1575,1576	.000000002	.105127377	19	30,-30
1577,1578	.000000002	.105127382	19	31,-31
1579,1580	.000000002	.105127386	19	32,-32

N	EIGEN VALUE	CUM. SUM.	P	Q
1581.1582	.000000002	.105127389	20	11,-11
1583.1584	.000000002	.105127393	19	33,-33
1585.1586	.000000002	.105127396	19	34,-34
1587.1588	.000000002	.105127399	19	35,-35
1589.1590	.000000001	.105127402	20	12,-12
1591.1592	.000000001	.105127405	19	36,-36
1593.1594	.000000001	.105127408	19	37,-37
1595.1596	.000000001	.105127410	19	38,-38
1597.1598	.000000001	.105127413	20	13,-13
1599.1600	.000000001	.105127415	19	39,-39
1601.1602	.000000001	.105127417	19	40,-40
1603.1604	.000000001	.105127419	19	41,-41
1605.1606	.000000001	.105127421	20	14,-14
1607.1608	.000000001	.105127423	20	15,-15
1609.1610	.000000001	.105127424	20	16,-16
1611.1612	.000000001	.105127425	20	17,-17
1613.1614	.000000000	.105127426	20	18,-18
1615.1616	.000000000	.105127427	20	19,-19
1617.1618	.000000000	.105127428	20	20,-20
1619.1620	.000000000	.105127429	20	21,-21
1621.1622	.000000000	.105127429	20	22,-22
1623.1624	.000000000	.105127430	20	23,-23
1625.1626	.000000000	.105127430	20	24,-24
1627.1628	.000000000	.105127430	20	25,-25
1629.1630	.000000000	.105127431	20	26,-26
1631.1632	.000000000	.105127431	20	27,-27
1633.1634	.000000000	.105127431	20	28,-28
1635.1636	.000000000	.105127432	20	29,-29
1637.1638	.000000000	.105127432	20	30,-30
1639.1640	.000000000	.105127432	20	31,-31
1641.1642	.000000000	.105127432	20	32,-32
1643.1644	.000000000	.105127433	20	33,-33
1645.1646	.000000000	.105127433	20	34,-34
1647.1648	.000000000	.105127433	20	35,-35
1649.1650	.000000000	.105127433	20	36,-36
1651.1652	.000000000	.105127433	20	37,-37
1653.1654	.000000000	.105127433	20	38,-38
1655.1656	.000000000	.105127434	20	39,-39
1657.1658	.000000000	.105127434	20	40,-40
1659.1660	.000000000	.105127434	20	41,-41

Table IV

CENSORING System Probabilities

P_{CENSOR} is the probability that an aperture of diameter D/r_0 will, during a single short exposure, receive a wavefront whose rms distortion over the aperture (with tilt not considered a form of distortion) will be less than one radian.

D/r_0	P_{CENSOR}
2	0.986 ± 0.006
3	0.765 ± 0.005
4	0.334 ± 0.014
5	$(9.38 \pm 0.33) \times 10^{-2}$
6	$(1.915 \pm 0.084) \times 10^{-2}$
7	$(2.87 \pm 0.57) \times 10^{-3}$
10	$(1.07 \pm 0.48) \times 10^{-6}$
15	$(3.40 \pm 0.59) \times 10^{-15}$

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2. F. S. Acton, "Numerical Methods That Work," Harper and Row, New York 1970. Section 13.

3. Ibid. Section 4.

4. I. S. Gradshteyn and I. M. Ryzhik, "Tables of Integrals, Series and Products," Academic Press, New York 1965, Eq. 3.6.31(17).

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